

# THE KLEIN-GORDON EQUATION, THE HILBERT TRANSFORM, AND DYNAMICS OF GAUSS-TYPE MAPS

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**ABSTRACT.** A pair  $(\Gamma, \Lambda)$ , where  $\Gamma \subset \mathbb{R}^2$  is a locally rectifiable curve and  $\Lambda \subset \mathbb{R}^2$  is a *Heisenberg uniqueness pair* if an absolutely continuous finite complex-valued Borel measure supported on  $\Gamma$  whose Fourier transform vanishes on  $\Lambda$  necessarily is the zero measure. Here, absolute continuity is with respect to arc length measure. Recently, it was shown by Hedenmalm and Montes that if  $\Gamma$  is the hyperbola  $x_1 x_2 = M^2/(4\pi^2)$ , where  $M > 0$  is the mass, and  $\Lambda$  is the lattice-cross  $(\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z})$ , where  $\alpha, \beta$  are positive reals, then  $(\Gamma, \Lambda)$  is a Heisenberg uniqueness pair if and only if  $\alpha\beta M^2 \leq 4\pi^2$ . The Fourier transform of a measure supported on a hyperbola solves the one-dimensional Klein-Gordon equation, so the theorem supplies very thin uniqueness sets for a class of solutions to this equation. By rescaling, we may assume that the mass equals  $M = 2\pi$ , and then the above-mentioned theorem is equivalent to the following assertion: *the functions*

$$e^{i\pi\alpha m t}, \quad e^{-i\pi\beta n/t}, \quad m, n \in \mathbb{Z},$$

*span a weak-star dense subspace of  $L^\infty(\mathbb{R})$  if and only if  $0 < \alpha\beta \leq 1$ .* The proof involved ideas from Ergodic Theory. To be more specific, in the critical regime  $\alpha\beta = 1$ , the crucial fact was that the Gauss-type map  $t \mapsto -1/t$  modulo  $2\mathbb{Z}$  on  $[-1, 1]$  has an ergodic absolutely continuous invariant measure with infinite total mass. However, the case of the semi-axis  $\mathbb{R}_+$  as well as the holomorphic counterpart remained open. In this work, we completely solve these two problems. Both results can be stated in terms of Heisenberg uniqueness, but here, we prefer the concrete formulation. As for the semi-axis, we show that the restriction to  $\mathbb{R}_+$  of the functions

$$e^{i\pi\alpha m t}, \quad e^{-i\pi\beta n/t}, \quad m, n \in \mathbb{Z},$$

*span a weak-star dense subspace of  $L^\infty(\mathbb{R}_+)$  if and only if  $0 < \alpha\beta < 4$ .* Moreover, in the critical regime  $\alpha\beta = 4$ , the weak-star span misses the mark by one dimension only. The proof is based on the dynamics of the standard Gauss map  $t \mapsto 1/t \bmod \mathbb{Z}$  on the interval  $[0, 1]$ . In particular, we find that for  $1 < \alpha\beta < 4$ , there exist nontrivial functions  $f \in L^1(\mathbb{R})$  with

$$\int_{\mathbb{R}} e^{i\pi\alpha m t} f(t) dt = \int_{\mathbb{R}} e^{-i\pi\beta n/t} f(t) dt = 0, \quad m, n \in \mathbb{Z},$$

and that each such function is uniquely determined by its restriction to any of the semiaxes  $\mathbb{R}_+$  and  $\mathbb{R}_-$ . This is an instance of *dynamical unique continuation*.

As for the holomorphic counterpart, we show that the functions

$$e^{i\pi\alpha m t}, \quad e^{-i\pi\beta n/t}, \quad m, n \in \mathbb{Z}_+ \cup \{0\},$$

*span a weak-star dense subspace of  $H_+^\infty(\mathbb{R})$  if and only if  $0 < \alpha\beta \leq 1$ .* Here,  $H_+^\infty(\mathbb{R})$  is the subspace of  $L^\infty(\mathbb{R})$  which consists of those functions whose Poisson extensions to the upper half-plane are holomorphic. In the critical regime  $\alpha\beta = 1$ , the proof relies on the nonexistence of a certain invariant distribution for the above-mentioned Gauss-type map on the interval  $] -1, 1[$ , which is a new result of dynamical flavor. The latter result is explained in full detail elsewhere.

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## 1. INTRODUCTION

**1.1. Heisenberg uniqueness pairs.** Let  $\mu$  be a finite complex-valued Borel measure in the plane  $\mathbb{R}^2$ , and associate with it the Fourier transform

$$\hat{\mu}(\xi) := \int_{\mathbb{R}^2} e^{i\pi\langle x, \xi \rangle} d\mu(x),$$

where  $x = (x_1, x_2)$  and  $\xi = (\xi_1, \xi_2)$ , with inner product

$$\langle x, \xi \rangle = x_1 \xi_1 + x_2 \xi_2.$$

The Fourier transform  $\hat{\mu}$  is a continuous and bounded function on  $\mathbb{R}^2$ . In [16], the concept of a Heisenberg uniqueness pair (HUP) was introduced. It is similar to the notion of weakly mutually annihilating pairs of Borel measurable sets having positive area measure, which appears, e.g., in the book by Havin and Jöricke [15]. For  $\Gamma \subset \mathbb{R}^2$  which is finite disjoint union of smooth curves in  $\mathbb{R}^2$ , let  $M(\Gamma)$  denote the Banach space of complex-valued finite Borel measures in  $\mathbb{R}^2$ , supported on  $\Gamma$ . Moreover, let  $AC(\Gamma)$  denote the closed subspace of  $M(\Gamma)$  consisting of the measures that are absolutely continuous with respect to arc length measure on  $\Gamma$ .

**Definition 1.1.1.** Let  $\Gamma$  be a finite disjoint union of smooth curves in  $\mathbb{R}^2$ . For a set  $\Lambda \subset \mathbb{R}^2$ , we say that  $(\Gamma, \Lambda)$  is a Heisenberg uniqueness pair provided that

$$\forall \mu \in AC(\Gamma) : \quad \hat{\mu}|_{\Lambda} = 0 \implies \mu = 0.$$

Heisenberg uniqueness pairs in which  $\Gamma$  is a straight line or the union of two parallel lines were described in [16]. Later, Blasi [5] solved particular cases of the union of three parallel lines. The ellipse case was considered independently by Lev and Sjölin in [21] and [27]; Sjölin also considered the parabola in [28]. More recently, Jaming and Kellay in [19] developed new tools to study Heisenberg uniqueness pairs for a variety of curves  $\Gamma$ .

**1.2. The Zariski closure.** We turn to the notion of the Zariski closure. Note that the Zariski topology (or hull-kernel topology) is a standard concept in e.g. Algebraic Geometry, in the setting of spaces of polynomials. As for notation, we let  $AC(\Gamma; \Lambda)$  be the subspace of  $AC(\Gamma)$  consisting of those measures  $\mu$  whose Fourier transform vanishes on  $\Lambda$ .

**Definition 1.2.1.** Let  $\Gamma$  be a finite disjoint union of smooth curves in  $\mathbb{R}^2$ , and let  $\Lambda \subset \mathbb{R}^2$  be arbitrary. With respect to  $AC(\Gamma)$ , the Zariski closure of  $\Lambda$  is the set

$$z\text{clos}_{\Gamma}(\Lambda) := \{\xi \in \mathbb{R}^2 : [\forall \mu \in AC(\Gamma; \Lambda) : \hat{\mu}(\xi) = 0]\}.$$

Less formally, the Zariski closure (or hull) is the set where the Fourier transform of a measure  $\mu \in AC(\Gamma)$  must vanish given that it already vanishes on  $\Lambda$ . Now, as the Fourier image of  $AC(\Gamma)$  does not form an algebra with respect to pointwise multiplication of functions, we cannot expect the Zariski closure to correspond to a topology. This means that the intersection of two Zariski closures need not be a closure itself. It is easy to see that the closure operation is idempotent, however:  $z\text{clos}_{\Gamma}^2 = z\text{clos}_{\Gamma}$ . In terms of the Zariski closure, we may express the uniqueness pair property conveniently:  $(\Gamma, \Lambda)$  is a Heisenberg uniqueness pair if and only if

$$z\text{clos}_{\Gamma}(\Lambda) = \mathbb{R}^2.$$

**1.3. The Klein-Gordon equation.** In natural units, the Klein-Gordon equation in one spatial dimension reads

$$\partial_t^2 u - \partial_x^2 u + M^2 u = 0.$$

In terms of the (preferred) coordinates

$$\xi_1 := t + x, \quad \xi_2 := t - x,$$

the Klein-Gordon equation becomes

$$(1.3.1) \quad \partial_{\xi_1} \partial_{\xi_2} u + \frac{M^2}{4} u = 0.$$

*Remark 1.3.1.* Since  $t^2 - x^2 = \xi_1 \xi_2$ , the *time-like vectors* (those vectors  $(t, x) \in \mathbb{R}^2$  with  $t^2 - x^2 > 0$ ) correspond to the union of the first quadrant  $\xi_1, \xi_2 > 0$  and the third quadrant  $\xi_1, \xi_2 < 0$  in the  $(\xi, \xi_2)$ -plane). Likewise, the *space-like vectors* correspond to the union of the second quadrant  $\xi_1 > 0, \xi_2 < 0$  and the fourth quadrant  $\xi_1 < 0, \xi_2 > 0$ .

**1.4. Fourier analytic treatment of the Klein-Gordon equation.** In the sequel, we will not need to talk about the time and space coordinates  $(t, x)$  as such. So, e.g., we are free to use the notation  $x = (x_1, x_2)$  for the Fourier dual coordinate to  $\xi = (\xi_1, \xi_2)$ .

Let  $\mathcal{M}(\mathbb{R}^2)$  denote the Banach space of all finite complex-valued Borel measures in  $\mathbb{R}^2$ . We suppose that  $u$  is the Fourier transform of a  $\mu \in \mathcal{M}(\mathbb{R}^2)$ :

$$(1.4.1) \quad u(\xi) = \hat{\mu}(\xi) := \int_{\mathbb{R}^2} e^{i\pi \langle x, \xi \rangle} d\mu(x), \quad \xi \in \mathbb{R}^2.$$

The assumption that  $u$  solves the Klein-Gordon equation (1.3.1) would ask that

$$\left(x_1 x_2 - \frac{M^2}{4\pi^2}\right) d\mu(x) = 0$$

as a measure on  $\mathbb{R}^2$ , which we see is the same as a requirement on the support set of the measure  $\mu$ :

$$(1.4.2) \quad \text{supp } \mu \subset \Gamma_M := \left\{x \in \mathbb{R}^2 : x_1 x_2 = \frac{M^2}{4\pi^2}\right\}.$$

The set  $\Gamma_M$  is a hyperbola. We may use the  $x_1$ -axis to supply a global coordinate for  $\Gamma_M$ , and define a complex-valued finite Borel measure  $\pi_1 \mu$  on  $\mathbb{R}$  by setting

$$(1.4.3) \quad \pi_1 \mu(E) = \int_E d\pi_1 \mu(x_1) := \mu(E \times \mathbb{R}) = \int_{E \times \mathbb{R}} d\mu(x).$$

We shall at times refer to  $\pi_1 \mu$  as the *compression* of  $\mu$  to the  $x_1$ -axis. It is easy to see that  $\mu$  may be recovered from  $\pi_1 \mu$ ; indeed,

$$(1.4.4) \quad u(\xi) = \hat{\mu}(\xi) = \int_{\mathbb{R}^\times} e^{i\pi[\xi_1 t + M^2 \xi_2 / (4\pi^2 t)]} d\pi_1 \mu(t), \quad \xi \in \mathbb{R}^2.$$

Here, we use the standard notational convention  $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$ . We note that  $\mu$  is absolutely continuous with respect to arc length measure on  $\Gamma_M$  if and only if  $\pi_1 \mu$  is absolutely continuous with respect to Lebesgue length measure on  $\mathbb{R}^\times$ .

**1.5. The lattice-cross as a uniqueness set for solutions to the Klein-Gordon equation.** For positive reals  $\alpha, \beta$ , let  $\Lambda_{\alpha, \beta}$  denote the lattice-cross

$$(1.5.1) \quad \Lambda_{\alpha, \beta} := (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}),$$

so that the spacing along the  $\xi_1$ -axis is  $\alpha$ , and along the  $\xi_2$ -axis it is  $\beta$ . In the work [16], Hedenmalm and Montes-Rodríguez found the following.

**Theorem 1.5.1.** (Hedenmalm, Montes) *Fix positive reals  $M, \alpha, \beta$ . Then  $(\Gamma_M, \Lambda_{\alpha, \beta})$  is a Heisenberg uniqueness pair if and only if  $\alpha\beta M^2 \leq 4\pi^2$ .*

In terms of the Zariski closure, the theorem says that

$$\text{zclos}_{\Gamma_M}(\Lambda_{\alpha, \beta}) = \mathbb{R}^2$$

holds if and only if  $\alpha\beta M^2 \leq 4\pi^2$ . By taking the relation (1.4.4) into account, and by reducing the redundancy of the constants (i.e., we may without loss of generality consider  $M = 2\pi$  and  $\alpha = 1$  only), Theorem 1.5.1 is equivalent to the following statement: *the linear span of the functions*

$$e^{i\pi m t}, \quad e^{-i\pi \beta n / t}, \quad m, n \in \mathbb{Z},$$

*is weak-star dense in  $L^\infty(\mathbb{R})$  if and only if  $\beta \leq 1$ .* Here, we supply new and unexpected insight into the theory of Heisenberg uniqueness pairs, such as a new connection with the standard Gauss map (motivated by Theorem 1.6.1), and, more importantly, we uncover, in the framework of

Fourier Analysis, profound connections between the Hilbert transform and the dynamics of transfer operators intimately related to Gauss-type maps leading up to Theorem 1.8.2.

**1.6. Dynamic unique continuation from a branch of the hyperbola.** Just looking at Theorem 1.5.1, one immediately is led to ask what happens if we replace the hyperbola  $\Gamma_M$  by one of its two branches, say

$$(1.6.1) \quad \Gamma_M^+ := \Gamma_M \cap (\mathbb{R}_+ \times \mathbb{R}_+) = \left\{ x \in \mathbb{R}^2 : x_1 x_2 = \frac{M^2}{4\pi^2} \text{ and } x_1 > 0 \right\}.$$

First, we will provide a uniqueness theorem for the branch  $\Gamma_M^+$  of the hyperbola  $\Gamma_M$ , which turns out to be closely related to the famous Gauss-Kuzmin-Wirsing operator and the Gauss map  $x \mapsto 1/x \bmod \mathbb{Z}$ .

**Theorem 1.6.1.** *Fix positive reals  $\alpha, \beta, M$ . Then  $(\Gamma_M^+, \Lambda_{\alpha, \beta})$  is a Heisenberg uniqueness pair if and only if  $\alpha\beta M^2 < 16\pi^2$ . Moreover, in the critical case  $\alpha\beta M^2 = 16\pi^2$ , the space  $\text{AC}(\Gamma_M^+, \Lambda_{\alpha, \beta})$  is the one-dimensional space spanned by the measure  $\mu_0 \in \text{AC}(\Gamma_M^+, \Lambda_{\alpha, \beta})$  whose  $x_1$ -compression is given by*

$$d\pi_1 \mu_0(t) := \left\{ \frac{1_{[0, 2/\alpha]}(t)}{2(2 + \alpha t)} - \frac{1_{[2/\alpha, +\infty]}(t)}{\alpha t(2 + \alpha t)} \right\} dt.$$

The proof of Theorem 1.6.1 is presented in Section 6. In the same section, it is also shown that in the critical parameter regime  $\alpha\beta = 16\pi^2$ , the couple  $(\Gamma_M^+, \Lambda_{\alpha, \beta}^*)$  is indeed a Heisenberg uniqueness pair, where  $\Lambda_{\alpha, \beta}^* := \Lambda_{\alpha, \beta} \cup \{\xi^*\}$ , and  $\xi^* \in (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$  is any point off the lattice-cross  $\Lambda_{\alpha, \beta}$  (see Theorem 6.1.1). The analysis of the proof of Theorem 6.1.1 involves a geometric object known as the *Nielsen spiral*.

Again, by taking the relation (1.4.4) into account, and by reducing the redundancy of the constants (i.e., we may without loss of generality consider  $M = 2\pi$  and  $\alpha = 1$  only), it is easy to see that Theorem 1.6.1 entails the following assertion: *the restriction to  $\mathbb{R}_+$  of the linear span of the functions*

$$e^{i\pi m t}, \quad e^{-i\pi \beta n/t}, \quad m, n \in \mathbb{Z},$$

*is weak-star dense in  $L^\infty(\mathbb{R}_+)$  if and only if  $\beta < 4$ . Moreover, if  $\beta = 4$  the weak-star closure of this linear span has codimension one in  $L^\infty(\mathbb{R}_+)$ .*

Theorem 1.6.1 has the following consequence in terms of unique continuation from the branch  $\Gamma_M^+$ , or the complementary branch  $\Gamma_M^- := \Gamma_M \setminus \Gamma_M^+$ , to the entire hyperbola  $\Gamma_M$ .

**Corollary 1.6.2.** *Fix positive reals  $\alpha, \beta, M$ . Then  $\mu \in \text{AC}(\Gamma_M, \Lambda_{\alpha, \beta})$  is uniquely determined by its restriction to the hyperbola branch  $\Gamma_M^-$  if and only if  $\alpha\beta M^2 < 16\pi^2$ . The same holds with  $\Gamma_M^-$  replaced by  $\Gamma_M^+$  as well.*

**1.7. The Zariski closure of the axes and half-axes.** We first consider the Zariski closure of the two axes  $\mathbb{R} \times \{0\}$  and  $\{0\} \times \mathbb{R}$  with respect to the space  $\text{AC}(\Gamma_M)$  of absolutely continuous measures, with respect to arc length, on the hyperbola  $\Gamma_M$ .

**Proposition 1.7.1.** *Fix a positive real  $M$ . If  $\mu \in \text{AC}(\Gamma_M)$  is such that  $\hat{\mu}$  vanishes one of the axes,  $\mathbb{R} \times \{0\}$  or  $\{0\} \times \mathbb{R}$ , then  $\mu = 0$  identically. In terms of Zariski closures, this means that*

$$\text{zcl}_{\Gamma_M}(\mathbb{R} \times \{0\}) = \text{zcl}_{\Gamma_M}(\{0\} \times \mathbb{R}) = \mathbb{R}^2.$$

The proof of Proposition 1.7.1 is supplied in Section 2.

The next proposition will show the difference between time-like and space-like quarter-planes. First, we need some notation. Let  $\mathbb{R}_+ := \{t \in \mathbb{R} : t > 0\}$  and  $\mathbb{R}_- := \{t \in \mathbb{R} : t < 0\}$  be the positive and negative half-lines, respectively. We write  $\bar{\mathbb{R}}_+ := \{t \in \mathbb{R} : t \geq 0\}$  and  $\bar{\mathbb{R}}_- := \{t \in \mathbb{R} : t \leq 0\}$  for the corresponding closed half-lines.

**Proposition 1.7.2.** *Fix a positive real  $M$ . Then the Zariski closures of each of the four semi-axes  $\mathbb{R}_+ \times \{0\}$ ,  $\mathbb{R}_- \times \{0\}$ ,  $\{0\} \times \mathbb{R}_+$ , and  $\{0\} \times \mathbb{R}_-$ , are as follows:*

$$\text{zcl}_{\Gamma_M}(\mathbb{R}_+ \times \{0\}) = \text{zcl}_{\Gamma_M}(\{0\} \times \mathbb{R}_-) = \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-$$

and

$$\text{zcl}_{\Gamma_M}(\mathbb{R}_- \times \{0\}) = \text{zcl}_{\Gamma_M}(\{0\} \times \mathbb{R}_+) = \bar{\mathbb{R}}_- \times \bar{\mathbb{R}}_+.$$

The proof of Proposition 1.7.2 is also supplied in Section 2.

*Remark 1.7.3.* In each of the instances in Proposition 1.7.2, we note that the Zariski closure of a semi-axis equals the topological closure of the adjacent quadrant of *space-like vectors*.

### 1.8. The Zariski closure of the lattice-cross restricted to a time-like or space-like quadrant.

Let us write

$$\mathbb{Z}_+ := \{1, 2, 3, \dots\}, \quad \mathbb{Z}_- := \{-1, -2, -3, \dots\}, \quad \mathbb{Z}_{+,0} := \{0, 1, 2, \dots\}, \quad \mathbb{Z}_{-,0} := \{0, -1, -2, \dots\}$$

for the sets of positive, negative, nonnegative, and nonpositive integers, respectively. We consider the following four portions of the lattice-cross  $\Lambda_{\alpha,\beta}$  given by (1.5.1):

$$\Lambda_{\alpha,\beta}^{++} := (\alpha\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_+), \quad \Lambda_{\alpha,\beta}^{+-} := (\alpha\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_-),$$

and

$$\Lambda_{\alpha,\beta}^{-+} := (\alpha\mathbb{Z}_{-,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_+), \quad \Lambda_{\alpha,\beta}^{--} := (\alpha\mathbb{Z}_{-,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_-).$$

We first calculate the Zariski closure of the two of these, that is, the first and third quadrants, which are time-like.

**Theorem 1.8.1.** (time-like) *Fix positive reals  $\alpha, \beta, M$ . Then for each point  $\xi^* \in \mathbb{R}^2 \setminus \Lambda_{\alpha,\beta}^{++}$ , there exists a measure  $\mu \in \text{AC}(\Gamma_M)$  such that  $\hat{\mu} = 0$  on  $\Lambda_{\alpha,\beta}^{++}$ , while at the same time  $\hat{\mu}(\xi^*) \neq 0$ . Moreover, the same assertion holds provided that  $\Lambda_{\alpha,\beta}^{++}$  is replaced by  $\Lambda_{\alpha,\beta}^{--}$ . In terms of Zariski closures, this means that*

$$\text{zcl}_{\Gamma_M}(\Lambda_{\alpha,\beta}^{++}) = \Lambda_{\alpha,\beta}^{++}, \quad \text{zcl}_{\Gamma_M}(\Lambda_{\alpha,\beta}^{--}) = \Lambda_{\alpha,\beta}^{--}.$$

The proof of Theorem 1.8.1, which is presented in Section 5, requires careful handling of the  $H^1$ -BMO duality and the explicit calculation of the Fourier transform of the unimodular function  $t \mapsto e^{i/t}$  as a tempered distribution.

We turn to the Zariski closures of the remaining two portions of the lattice-cross. We first write down the statement in terms of weak-star closure of the linear span of a sequence of unimodular functions, and then explain what it means for the Zariski closure in the form of a corollary. This is our second main result.

As for notation, let  $H_+^\infty(\mathbb{R})$  denote the weak-star closed subspace of  $L^\infty(\mathbb{R})$  consisting of those functions whose Poisson extension to the upper half-plane is holomorphic.

**Theorem 1.8.2.** *Fix positive reals  $\alpha, \beta$ . Then the functions*

$$e^{i\pi\alpha m t}, \quad e^{-i\pi\beta n/t}, \quad m, n = 0, 1, 2, \dots,$$

*which are elements of  $H_+^\infty(\mathbb{R})$ , span together a weak-star dense subspace of  $H_+^\infty(\mathbb{R})$  if and only if  $\alpha\beta \leq 1$ .*

A standard Möbius mapping brings the upper half-plane to the unit disk  $\mathbb{D}$ , and identifies the space  $H_+^\infty(\mathbb{R})$  with  $H^\infty(\mathbb{D})$ , the space of all bounded holomorphic functions on  $\mathbb{D}$ . For this reason, Theorem 1.8.2 is equivalent to the following assertion, which we state as a corollary.

**Corollary 1.8.3.** *Fix positive reals  $\lambda_1, \lambda_2$ . Then the linear span of the inner functions*

$$\phi_1(z)^m = \exp\left(m\lambda_1 \frac{z+1}{z-1}\right) \quad \text{and} \quad \phi_2(z)^n = \exp\left(n\lambda_2 \frac{z-1}{z+1}\right), \quad m, n = 0, 1, 2, \dots,$$

*is weak-star dense set in  $H^\infty(\mathbb{D})$  if and only if  $\lambda_1\lambda_2 \leq \pi^2$ .*

We suppress the trivial proof of the corollary.

*Remark 1.8.4.* Clearly, Corollary 1.8.3 supplies a complete and affirmative answer to Problems 1 and 2 in [23]. We recall the question from [23]: the issue was raised whether the algebra generated by the two inner functions

$$\phi_1(z) = \exp\left(\lambda_1 \frac{z+1}{z-1}\right) \quad \text{and} \quad \phi_2(z) = \exp\left(\lambda_2 \frac{z-1}{z+1}\right)$$

for  $0 < \lambda_1, \lambda_2 < +\infty$ , is weak-star dense in  $H^\infty(\mathbb{D})$  if and only if  $\lambda_1 \lambda_2 \leq \pi^2$ . The “only if” was understood already in [23]. As pointed out in [23], it is a consequence of Corollary 1.8.3 that for  $\lambda_1 \lambda_2 \leq \pi^2$ , the lattice of the closed subspaces invariant with respect to multiplication by the two inner functions  $\phi_1, \phi_2$  coincides with the usual shift invariant subspaces in the Hardy space  $H^p(\mathbb{D})$ , where  $1 < p < +\infty$ .

*Remark 1.8.5.* It is impossible to derive the assertion of Theorem 1.8.2 from Theorem 1.5.1. It is a *much finer statement*. In Section 11, we explain how the result relies on a hitherto unknown result, presented in [17], which extends the standard ergodic theory for certain Gauss-type transformations on the interval  $I_1 := ]-1, 1[$ , where the novelty is that we may handle distributions where the standard theory has only measures. The relevant space of distributions is obtained as the restriction to  $I_1$  of  $L^1(\mathbb{R})$  plus  $\mathbf{H}L^1(\mathbb{R})$ , where  $\mathbf{H}$  is the Hilbert transform (i.e., convolution with the principal value distribution  $\text{pv} \frac{1}{\pi t}$  on the line). The issue has to do with the uniqueness of the absolutely continuous invariant measure in the larger space. Thinking physically, in the larger space, we have two types of particles, localized and delocalized. The localized particles are represented by  $\delta_\xi$ , whereas delocalized particles are represented by  $\mathbf{H}\delta_\xi$ , for some real  $\xi$ . The state space allows for scalar multiples of localized and delocalized particles, and linear combinations of them. Finally, we are looking for such localized and delocalized particles smeared out in an absolutely continuous way, and call it an *invariant state* if it is preserved under the corresponding Gauss-type map. This generalizes the notion of the absolutely continuous invariant measure which is standard in ergodic theory, and since uniqueness issues for the invariant measure translate to ergodic properties, we are left with a far-reaching generalization of ergodic theory. We have not been able to find any appropriate references for similar considerations in the literature, but suggest that there may be some relevance of the works [3] and [4] for the discrete setting, and [6] for flows.

All the effort in [17] is developed to deal with the “if” part of the assertion of Theorem 1.8.2. On the other hand, the “only if” part is much simpler, as for instance the work in [7] shows that in case  $\alpha\beta > 1$ , the weak-star closure of the linear span in question has infinite codimension in  $H_+^\infty(\mathbb{R})$ .

Theorem 1.8.2 can be restated in terms of uniqueness properties of solutions to the Klein-Gordon equation. Note that in the statement below, the pair  $(\Lambda_{\alpha,\beta}^{+-}, \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-)$  can be replaced by  $(\Lambda_{\alpha,\beta}^{-+}, \bar{\mathbb{R}}_- \times \bar{\mathbb{R}}_+)$  without perturbing the validity of the result.

**Corollary 1.8.6.** *Fix positive reals  $\alpha, \beta, M$  with  $\alpha\beta M^2 \leq 4\pi^2$ . Suppose that  $u = \hat{\mu}$  solves the Klein-Gordon equation (1.3.1), where  $\mu$  is finite complex Borel measure on  $\mathbb{R}^2$ , which is assumed absolutely continuous with respect to one-dimensional Hausdorff measure. Then the values of  $u$  on the space-like quarter-plane  $\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-$  are determined by the values of  $u$  on the set  $\Lambda_{\alpha,\beta}^{+-}$ , which is the portion of the lattice-cross in the given quarter-plane. This property does not hold for  $\alpha\beta M^2 > 4\pi^2$ .*

This formulation is actually a consequence of the Zariski closure result of Corollary 1.8.7 below, so we refer to the explanatory remarks that follow right after it.

**Corollary 1.8.7.** (space-like) *Fix positive reals  $\alpha, \beta, M$ . The following assertions are equivalent:*

- (i)  $\text{zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}^{+-}) = \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-$ ,
- (ii)  $\text{zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}^{-+}) = \bar{\mathbb{R}}_- \times \bar{\mathbb{R}}_+$ ,
- (iii)  $\alpha\beta M^2 \leq 4\pi^2$ .

Here, the main part of the equivalence (i) $\Leftrightarrow$ (iii) is the implication (iii) $\Rightarrow$ (i'), where (i') is as follows:

$$(i') \text{ zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}^{+-}) \supset \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-.$$

The latter implication can be understood in the following terms. Under the density condition (iii), any measure  $\mu \in \text{AC}(\Gamma_M)$  whose Fourier transform  $\hat{\mu}$  vanishes on  $\Lambda_{\alpha,\beta}^{+-}$ , has the property that  $\hat{\mu}$  actually vanishes on the entire space-like adjacent quarter-plane  $\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-$ . This assertion is seen to be equipotent with Theorem 1.8.2, after a scaling argument which permits us to assume that  $M := 2\pi$ . Finally, to obtain the equality (i) from the inclusion (i') which results from Theorem 1.8.2, we may use e.g. Proposition 1.7.2. The remaining equivalence (ii) $\Leftrightarrow$ (iii) is, by a symmetry argument, the same as the the equivalence (i) $\Leftrightarrow$ (iii).

*Remark 1.8.8.* Let us now explain how Theorem 1.5.1 is an immediate consequence of the much deeper result of Corollary 1.8.7. First, an elementary argument (see [16], [7]) shows that  $\text{zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}) \neq \mathbb{R}^2$  for  $\alpha\beta M^2 > 4\pi^2$ , so that we just need to obtain the implication

$$\alpha\beta M^2 \leq 4\pi^2 \implies \text{zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}) = \mathbb{R}^2.$$

In view of Theorem 1.8.2,

$$\alpha\beta M^2 \leq 4\pi^2 \implies \text{zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}) = \text{zclos}_{\Gamma_M}(\Lambda_{\alpha,\beta}^{+-} \cup \Lambda_{\alpha,\beta}^{-+}) \supset (\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-) \cup (\bar{\mathbb{R}}_- \times \bar{\mathbb{R}}_+) \supset \mathbb{R} \times \{0\},$$

and Theorem 1.5.1 becomes a consequence of Proposition 1.7.1 together with the idempotent property  $\text{zclos}_{\Gamma}^2 = \text{zclos}_{\Gamma}$ .

We now turn to the underlying ideas connected with the dynamics of Gauss-type maps and the Hilbert transform.

## 2. THE ZARISKI CLOSURES OF THE AXES OR SEMI-AXES

**2.1. The standard Hardy spaces  $H_+^p(\mathbb{R})$ .** The Hardy space  $H_+^\infty(\mathbb{R})$  consists of all functions  $f \in L^\infty(\mathbb{R})$  with Poisson extension to the upper half-plane

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \text{Im } z > 0\}$$

which is holomorphic. Here, the Poisson extension of  $f$  is given by the expression

$$f(z) := \frac{\text{Im } z}{\pi} \int_{\mathbb{R}} \frac{f(t)}{|z - t|^2} dt, \quad z \in \mathbb{C}_+.$$

In a similar fashion, for  $1 \leq p < +\infty$ , we say that  $f \in H_+^p(\mathbb{R})$  if  $f \in L^p(\mathbb{R})$  and its Poisson extension is holomorphic in  $\mathbb{C}_+$ .

**2.2. The Zariski closures of the axes and semi-axes.** We now supply the proofs of Propositions 1.7.1 and 1.7.2.

*Proof of Proposition 1.7.1.* By symmetry, it is enough to show that  $\text{zclos}_{\Gamma_M}(\mathbb{R} \times \{0\}) = \mathbb{R}^2$ . More concretely, we need to show that if  $\mu \in \text{AC}(\Gamma_M)$  and

$$\hat{\mu}(\xi_1, 0) = 0, \quad \xi_1 \in \mathbb{R},$$

then  $\mu = 0$  as a measure. In view of (1.4.4),

$$\hat{\mu}(\xi_1, 0) = \int_{\mathbb{R}^\times} e^{i\pi\xi_1 t} d\pi_1\mu(t),$$

where  $\pi_1\mu$  is the compression of  $\mu$  to the real line. The uniqueness theorem for the Fourier transform gives that  $\pi_1\mu = 0$ , and hence that  $\mu = 0$ , since  $\mu$  and its compression  $\pi_1\mu$  are in a one-to-one correspondence.  $\square$

*Proof of Proposition 1.7.2.* By symmetry, it is enough to show that

$$\text{zclos}_{\Gamma_M}(\mathbb{R}_+ \times \{0\}) = \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-$$

To this end, we consider a measure  $\mu \in \text{AC}(\Gamma_M)$  with (use (1.4.4))

$$\hat{\mu}(\xi_1, 0) = \int_{\mathbb{R}} e^{i\pi\xi_1 t} d\pi_1\mu(t) = 0, \quad \xi_1 \in \mathbb{R}_+.$$

This condition is equivalent to asking that  $d\pi_1\mu(t) = f(t)dt$ , where  $f \in H_+^1(\mathbb{R})$ . It follows from standard arguments that

$$\int_{\mathbb{R}} g(t) d\pi_1\mu(t) = \int_{\mathbb{R}} f(t)g(t)dt = 0$$

for all  $g \in H_+^\infty(\mathbb{R})$ . We observe that for  $\xi_1 \geq 0$  and  $\xi_2 \leq 0$ , the function

$$g(t) := e^{i\pi[\xi_1 t + M^2 \xi_2 / (4\pi^2 t)]}$$

is in  $H_+^\infty(\mathbb{R})$ , and so

$$\hat{\mu}(\xi_1, \xi_2) = \int_{\mathbb{R}^\times} e^{i\pi[\xi_1 t + M^2 \xi_2 / (4\pi^2 t)]} d\pi_1\mu(t) = 0, \quad (\xi_1, \xi_2) \in \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-.$$

In conclusion, this argument proves the inclusion

$$\text{zclos}_{\Gamma_M}(\mathbb{R}_+ \times \{0\}) \supset \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-.$$

To obtain the equality of the two sides, we need to show that if  $(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus (\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-)$ , then there exists a  $\mu \in \text{AC}(\Gamma_M)$  with  $d\pi_1\mu(t) = f(t)dt$ , where  $f \in H_+^1(\mathbb{R})$ , such that  $\hat{\mu}(\xi_1, \xi_2) \neq 0$ . But then the bounded function

$$g(t) = e^{i\pi[\xi_1 t + M^2 \xi_2 / (4\pi^2 t)]}, \quad t \in \mathbb{R},$$

is not an element of  $H_+^\infty(\mathbb{R})$ , and by the standard Hardy space duality theory,

$$\sup \left\{ \left| \int_{\mathbb{R}} f(t)g(t)dt \right| : f \in \text{ball}(H_+^1(\mathbb{R})) \right\} = \inf \{ \|g - h\|_{L^\infty(\mathbb{R})} : h \in H_+^\infty(\mathbb{R}) \} > 0.$$

In particular, there must exist an  $f \in H_+^1(\mathbb{R})$  with

$$\hat{\mu}(\xi_1, \xi_2) = \int_{\mathbb{R}} f(t)g(t)dt \neq 0.$$

This completes the proof.  $\square$

### 3. BASIC PROPERTIES OF THE DYNAMICS OF GAUSS-TYPE MAPS ON INTERVALS

**3.1. Notation for intervals.** For a positive real  $\gamma$ , let  $I_\gamma := ]-\gamma, \gamma[$  denote the corresponding symmetric open interval, and let  $I_\gamma^+ := ]0, \gamma[$  be the positive side of the interval  $I_\gamma$ . At times, we will need the half-open intervals  $\tilde{I}_\gamma := ]-\gamma, \gamma]$  and  $\tilde{I}_\gamma^+ := [0, \gamma[$ , as well as the closed intervals  $\bar{I}_\gamma := [-\gamma, \gamma]$  and  $\bar{I}_\gamma^+ := [0, \gamma]$ .

**3.2. Dual action notation.** For a Lebesgue measurable subset  $E$  of the real line  $\mathbb{R}$ , we write

$$\langle f, g \rangle_E := \int_E f(t)g(t)dt,$$

whenever  $fg \in L^1(E)$ . This will be of interest mainly when  $E$  is an open interval, and in this case, we use the same notation to describe the dual action of a distribution on a test function.



**3.3. Gauss-type maps on intervals.** For background material in Ergodic Theory, we refer to the book [8].

For  $x \in \mathbb{R}$ , let  $\{x\}_1$  denote the *fractional part of  $x$* , that is, the unique number in the half-open interval  $\tilde{I}_1^+ = [0, 1[$  with  $x - \{x\}_1 \in \mathbb{Z}$ . Likewise, we let  $\{x\}_2$  denote the *even-fractional part of  $x$* , by which we mean the unique number in the half-open interval  $\tilde{I}_1 = ]-1, 1]$  with  $x - \{x\}_2 \in 2\mathbb{Z}$ . We will be interested in the Gauss-type maps  $\sigma_\gamma : \tilde{I}_1^+ \rightarrow \tilde{I}_1^+$  and  $\tau_\beta : \tilde{I}_1 \rightarrow \tilde{I}_1$  given by

$$\sigma_\gamma(x) := \left\{ \frac{\gamma}{x} \right\}_1 \quad \text{and} \quad \tau_\beta(x) := \left\{ -\frac{\beta}{x} \right\}_2.$$

Here,  $\beta, \gamma$  are reals with  $0 < \beta, \gamma \leq 1$ . Then  $\sigma_1$  is the classical Gauss map of the unit interval  $I_1^+$ .

**3.4. Transfer, subtransfer, and compressed Koopman operators.** Fix two reals  $\beta, \gamma$  with  $0 < \beta, \gamma \leq 1$ . Let  $\mathbf{K}_\gamma : L^\infty(I_1^+) \rightarrow L^\infty(I_1^+)$  and  $\mathbf{L}_\beta : L^\infty(I_1) \rightarrow L^\infty(I_1)$  and denote the *compressed Koopman operators* (or *sub-Koopman operators*)

$$(3.4.1) \quad \mathbf{K}_\gamma f(x) := 1_{I_\gamma^+}(x) f \circ \sigma_\gamma(x), \quad \mathbf{L}_\beta f(x) := 1_{I_\beta}(x) f \circ \tau_\beta(x).$$

Here, as always,  $1_E$  stands for the characteristic function of the set  $E$ , which equals 1 on  $E$  and vanishes elsewhere. The *subtransfer operators*  $\mathbf{S}_\gamma : L^1(I_\gamma^+) \rightarrow L^1(I_\gamma^+)$  and  $\mathbf{T}_\beta : L^1(I_1) \rightarrow L^1(I_1)$  are defined by

$$(3.4.2) \quad \mathbf{S}_\gamma f(x) := \sum_{j=1}^{+\infty} \frac{\gamma}{(j+x)^2} f\left(\frac{\gamma}{j+x}\right), \quad \mathbf{T}_\beta f(x) := \sum_{j \in \mathbb{Z}^\times} \frac{\beta}{(2j+x)^2} f\left(-\frac{\beta}{2j+x}\right).$$

Here, we use the notation  $\mathbb{Z}^\times := \mathbb{Z} \setminus \{0\}$ . A standard argument shows that

$$(3.4.3) \quad \begin{cases} \langle \mathbf{S}_\gamma f, g \rangle_{I_1^+} = \langle f, \mathbf{K}_\gamma g \rangle_{I_1^+}, & f \in L^1(I_1^+), \quad g \in L^\infty(I_1^+), \\ \langle \mathbf{T}_\beta f, g \rangle_{I_1} = \langle f, \mathbf{L}_\beta g \rangle_{I_1}, & f \in L^1(I_1), \quad g \in L^\infty(I_1); \end{cases}$$

in other words,  $\mathbf{S}_\gamma$  is the preadjoint of  $\mathbf{K}_\gamma$ , and  $\mathbf{T}_\beta$  is the preadjoint of  $\mathbf{L}_\beta$ .

The *cone of positive functions* consists of all integrable functions  $f$  with  $f \geq 0$  a.e. on the respective interval. Similarly, we say that  $f$  is *positive* if  $f \geq 0$  a.e. on the given interval.

**Proposition 3.4.1.** Fix  $0 < \beta, \gamma \leq 1$ . The operators  $\mathbf{T}_\beta : L^1(I_1) \rightarrow L^1(I_1)$  and  $\mathbf{S}_\gamma : L^1(I_\gamma^+) \rightarrow L^1(I_\gamma^+)$  are both norm contractions, which preserve the respective cones of positive functions. For  $\beta = \gamma = 1$ ,  $\mathbf{T}_1$  and  $\mathbf{S}_1$  act isometrically on the positive functions. The associated adjoints  $\mathbf{L}_\beta : L^\infty(I_1) \rightarrow L^\infty(I_1)$  and  $\mathbf{K}_\gamma : L^\infty(I_1^+) \rightarrow L^\infty(I_1^+)$  are norm contractions as well.

This is well-known for  $\gamma = \beta = 1$  and very easy to obtain for  $0 < \beta, \gamma < 1$ .

**3.5. An elementary observation and an estimate of the  $\mathbf{T}_\beta$ -orbit of certain functions.** We begin with the following elementary observation.

**OBSERVATION.** The subtransfer operators  $\mathbf{T}_\beta, \mathbf{S}_\gamma$ , initially defined on  $L^1$  functions, make sense for wider classes of functions. Indeed, if  $f \geq 0$ , then the formulae (3.4.2) make sense pointwise, with values in the extended nonnegative reals  $[0, +\infty]$ . More generally, if  $f$  is complex-valued, we may use the triangle inequality to dominate the convergence of  $\mathbf{T}_\beta f$  by that of  $\mathbf{T}_\beta |f|$ . This entails that  $\mathbf{T}_\beta f$  is well-defined a.e. if  $\mathbf{T}_\beta |f| < +\infty$  holds a.e. The same goes for  $\mathbf{S}_\gamma$  of course.

In view of the above observation, it is meaningful to try to control  $\mathbf{T}_\beta f$  for  $f \geq 0$ . The following basic size estimate is useful.

**Proposition 3.5.1.** Fix  $0 < \beta \leq 1$ . If  $f : I_1 \rightarrow \mathbb{R}$  is even and its restriction to  $I_1^+$  is increasing, and if  $f \geq 0$ , then

$$\beta C_0 f(0) \leq \mathbf{T}_\beta f(x) - \frac{\beta}{(2 - |x|)^2} f\left(\frac{\beta}{2 - |x|}\right) \leq \beta C_1 f\left(\frac{1}{2}\beta\right), \quad x \in I_1,$$

where  $C_0 := \frac{\pi^2}{6} - \frac{5}{4} = 0.3949 \dots$  and  $C_1 := \frac{\pi^2}{6} - 1 = 0.6449 \dots$

*Proof.* For convenience of notation, we write

$$(3.5.1) \quad s_j(x) := -\frac{\beta}{2j+x},$$

which is an increasing function on  $I_1$  for  $j \in \mathbb{Z}^\times = \mathbb{Z} \setminus \{0\}$ . We first consider the right half of the interval, i.e.,  $x \in I_1^+ = ]0, 1[$ . As  $f$  is even, we see that

$$f(s_j(x)) = f\left(-\frac{\beta}{2j+x}\right) = f\left(\frac{\beta}{2j+x}\right),$$

and since  $f$  is increasing on  $I_1^+$ , we obtain that for integers  $j \geq 1$ ,

$$f(0) \leq f\left(\frac{\beta}{2j+1}\right) \leq f(s_j(x)) = f\left(\frac{\beta}{2j+x}\right) \leq f\left(\frac{\beta}{2j}\right) \leq f\left(\frac{1}{2}\beta\right), \quad x \in I_1^+,$$

while for integers  $j \leq -2$  we have a similar estimate:

$$f(0) \leq f\left(\frac{\beta}{2|j|}\right) \leq f(s_j(x)) = f\left(\frac{\beta}{2|j|-x}\right) \leq f\left(\frac{\beta}{2|j|-1}\right) \leq f\left(\frac{1}{3}\beta\right) \leq f\left(\frac{1}{2}\beta\right), \quad x \in I_1^+.$$

Since

$$\mathbf{T}_\beta f(x) - \frac{\beta}{(2-x)^2} f\left(\frac{\beta}{2-x}\right) = \frac{1}{\beta} \sum_{j \in \mathbb{Z} \setminus \{0, -1\}} [s_j(x)]^2 f(s_j(x)),$$

the claimed estimate follows from

$$\frac{\pi^2}{6} - \frac{5}{4} \leq \frac{1}{\beta^2} \sum_{j \in \mathbb{Z} \setminus \{0, -1\}} [s_j(x)]^2 \leq \frac{\pi^2}{6} - \frac{5}{4}, \quad x \in I_1^+.$$

The remaining case when  $x \in I_1^- := ]-1, 0[$  is analogous.  $\square$

**3.6. Symmetry preservation of the subtransfer operator  $\mathbf{T}_\beta$ .** The fact that the action of  $\mathbf{T}_\beta$  commutes with the reflection in the origin will be needed. The precise formulation reads as follows. Let  $\check{\mathbf{I}}$  be the antipodal operator  $\check{\mathbf{I}}f(x) := f(-x)$ , which is its own inverse:  $\check{\mathbf{I}}^2 = \mathbf{I}$ .

**Proposition 3.6.1.** *Fix  $0 < \beta \leq 1$ . Suppose  $f : I_1 \rightarrow \mathbb{R}$  is a function satisfying  $\mathbf{T}_\beta |f|(x) < +\infty$  for a point  $x \in I_1$ . Then*

$$\mathbf{T}_\beta f(x) = \check{\mathbf{I}} \mathbf{T}_\beta \check{\mathbf{I}} f(x).$$

*Proof.* We keep the notation  $s_j(x) = -\beta/(2j+x)$  from Proposition 3.5.1, and note that

$$s_{-j}(-x) = -s_j(x),$$

which gives that

$$\check{\mathbf{I}} \mathbf{T}_\beta \check{\mathbf{I}} f(x) = \frac{1}{\beta} \sum_{j \in \mathbb{Z}^\times} [s_{-j}(-x)]^2 f(-s_{-j}(-x)) = \frac{1}{\beta} \sum_{j \in \mathbb{Z}^\times} [s_j(x)]^2 f(s_j(x)) = \mathbf{T}_\beta f(x).$$

The assumption  $\mathbf{T}_\beta |f|(x) < +\infty$  guarantees the absolute convergence of the above series.  $\square$

**3.7. Symmetry, monotonicity, convexity, and the operator  $\mathbf{T}_\beta$ .** We may now derive the property that  $\mathbf{T}_\beta$  preserves the class of functions that are odd and increasing.

**Proposition 3.7.1.** *Fix  $0 < \beta \leq 1$ . If  $f : I_1 \rightarrow \mathbb{R}$  is odd and (strictly) increasing, then so is  $\mathbf{T}_\beta f : I_1 \rightarrow \mathbb{R}$ .*

*Proof.* If  $f$  is odd and increasing, then  $|f|$  is even and its restriction to  $I_1^+$  is increasing. From Proposition 3.5.1, we get that  $\mathbf{T}_\beta |f|(x) < +\infty$  for every  $x \in I_1$ , so that by Proposition 3.6.1,  $\mathbf{T}_\beta f(x) = -\mathbf{T}_\beta f(-x)$ , which means that  $\mathbf{T}_\beta f$  is odd. Since

$$\mathbf{T}_\beta f(x) = \frac{1}{\beta} \sum_{j \in \mathbb{Z}^\times} [s_j(x)]^2 f(s_j(x)) = \frac{1}{\beta} \sum_{j \in \mathbb{Z}^\times} t^2 f(t) \Big|_{t=s_j(x)},$$

where  $s_j(x) = -\beta/(2j+x)$  is known to be increasing on  $I_1$  for each  $j \in \mathbb{Z}^\times$ , it is enough to check that  $t^2 f(t)$  is increasing in  $t \in I_1$ , which in its turn is an immediate consequence of the assumption that  $f$  is odd and increasing. The strict case is analogous.  $\square$

We can now derive the property that  $\mathbf{T}_\beta$  preserves the class of functions that are positive, even, and convex.

**Proposition 3.7.2.** *Fix  $0 < \beta \leq 1$ . If  $f : I_1 \rightarrow \mathbb{R}$  is even and convex, and if  $f \geq 0$ , then so is  $\mathbf{T}_\beta f$ .*

*Proof.* From Proposition 3.5.1 we see that  $0 \leq \mathbf{T}_\beta f(x) < +\infty$  holds for each  $x \in I_1$ . We keep the notation  $s_j(x) = -\beta/(2j+x)$  from Proposition 3.5.1. Since  $f$  is even, we know from Proposition 3.6.1 that  $\mathbf{T}_\beta f$  is even as well. A direct calculation, based on  $s'_j(x) = \beta^{-1}[s_j(x)]^2$ , shows that

$$\frac{d}{dx} \{ [s_j(x)]^2 f(s_j(x)) \} = \frac{1}{\beta} \left( 2t^3 f(t) + t^4 f'(t) \right) \Big|_{t=s_j(x)}$$

where both sides are understood not in the pointwise but in the sense of distribution theory. Convexity means that the derivative is increasing, so we need to check that the left-hand side is increasing as a function of  $x$ . Now, since the function  $x \mapsto s_j(x)$  is increasing on  $I_1$  for each  $j \in \mathbb{Z}^\times$ , the above calculation gives that it is enough to check that the function  $t \mapsto 2t^3 f(t) + t^4 f'(t)$  is increasing on  $I_1$ . By assumption,  $f'(t)$  is odd and increasing, and hence  $t^4 f'(t)$  is odd and increasing too. Moreover, as  $f(t)$  is even and convex,  $f$  is increasing on  $I_1$ . Thus  $t \mapsto t^3 f(t)$  is odd and increasing on  $I_1$ . The statement now follows from the fact that the sum of convex functions is convex as well.  $\square$

**3.8. Preservation of continuous functions under  $\mathbf{T}_\beta$ .** For  $\gamma$  with  $0 < \gamma < +\infty$ , let  $C(\bar{I}_\gamma)$  denote the space of continuous functions on the compact symmetric interval  $\bar{I}_\gamma = [-\gamma, \gamma]$ . The following observation is immediate and hence its proof suppressed.

**Proposition 3.8.1.** *Fix  $0 < \beta \leq 1$ . If  $f \in C(\bar{I}_\beta)$ , then  $\mathbf{T}_\beta f \in C(\bar{I}_1)$ .*

**Proposition 3.8.2.** *Fix  $0 < \beta \leq 1$ . If  $f \in C(\bar{I}_\beta)$  is odd, then  $\mathbf{T}_\beta f(1) = \beta f(\beta)$ .*

*Proof.* By (3.4.2) and the assumption that  $f$  is odd, cancellation of all terms except for the one corresponding to index  $j = -1$  gives that

$$\mathbf{T}_\beta f(1) = \sum_{j \in \mathbb{Z}^\times} \frac{\beta}{(2j+1)^2} f\left(-\frac{\beta}{2j+1}\right) = \beta f(\beta).$$

The proof is complete.  $\square$

**3.9. Subinvariance of certain key functions.** It is well-known that the Gauss map  $\sigma_1(x) = \{1/x\}_1$  has the absolutely continuous invariant measure

$$\frac{dt}{(1+t) \log 2}, \quad t \in I_1^+,$$

normalized to be a probability measure. This suggests that we should analyze the behavior of the subtransfer operator  $\mathbf{S}_\gamma$  on the function

$$\lambda_1(x) := \frac{1}{1+x}, \quad x \in I_1^+.$$

**Proposition 3.9.1.** *Fix  $0 < \gamma \leq 1$ . With  $\lambda_1(x) = 1/(1+x)$  on  $I_1$ , we have that for  $n = 1, 2, 3, \dots$ ,*

$$\mathbf{S}_\gamma^n \lambda_1(x) \leq \left( \frac{2\gamma}{1+\gamma} \right)^n \lambda_1(x), \quad x \in I_1^+.$$

*Proof.* We first establish the assertion for  $n = 1$ . It is elementary to establish that for  $j = 1, 2, 3, \dots$ ,

$$\frac{\gamma}{(j+x)(j+x+\gamma)} \leq \frac{2\gamma}{1+\gamma} \frac{1}{(j+x)(j+x+1)}, \quad x \in I_1^+,$$

and since

$$\mathbf{S}_\gamma \lambda_1(x) = \sum_{j=1}^{+\infty} \frac{\gamma}{(j+x)^2} \frac{1}{1 + \frac{\gamma}{j+x}} = \sum_{j=1}^{+\infty} \frac{\gamma}{(j+x)(j+x+\gamma)},$$

the assertion of the proposition for  $n = 1$  now follows from the telescoping sum identity

$$\sum_{j=1}^{+\infty} \frac{1}{(j+x)(j+x+1)} = \sum_{j=1}^{+\infty} \left\{ \frac{1}{j+x} - \frac{1}{j+x+1} \right\} = \frac{1}{1+x}, \quad x \in I_1^+.$$

Finally, the assertion for  $n > 1$  follows by repeated application of the  $n = 1$  case, using that  $\mathbf{S}_\gamma$  is positive, i.e., it preserves the positive cone.  $\square$

Next, we consider the  $\mathbf{T}_\beta$ -iterates of the function (for  $0 < \alpha \leq 1$ )

$$(3.9.1) \quad \kappa_\alpha(x) := \frac{\alpha}{\alpha^2 - x^2}, \quad x \in I_1.$$

This function is not in  $L^1(I_1)$ , although it is in  $L^{1,\infty}(I_1)$ , the weak  $L^1$ -space; however, by the observation made in Subsection 3.5, we may still calculate the expression  $\mathbf{T}_\beta \kappa_\alpha$  pointwise wherever  $\mathbf{T}_\beta |\kappa_\alpha|(x) < +\infty$ . Note that  $\kappa_1(x)dx$  is the invariant measure for the transformation  $\tau_1(x) = \{-1/x\}_2$ , which in terms of the transfer operator  $\mathbf{T}_1$  means that  $\mathbf{T}_1 \kappa_1 = \kappa_1$ . It is of fundamental importance in most of our considerations that this invariant measure has *infinite mass*, i.e., that  $\kappa_1 \notin L^1(I_1)$ . The reason for this is that  $\tau_1$  has indifferent fixed points. The Gauss map  $\sigma_1$ , on the other hand, has only repelling fixed points, and an invariant measure  $\lambda_1(x)dx$  with finite mass. This is the main reason why the transfer operators  $\mathbf{S}_1$  and  $\mathbf{T}_1$  behave differently. We should add that the control of the orbits is much more difficult and not so well understood in the case of indifferent fixed points, in contrast with the case of repelling fixed points when the theory is well developed.

**Lemma 3.9.2.** Fix  $0 < \beta \leq 1$ . For the function  $\kappa_\beta(x) = \beta/(\beta^2 - x^2)$ , we have that

$$\mathbf{T}_\beta \kappa_\beta(x) = \mathbf{T}_\beta |\kappa_\beta|(x) = \kappa_1(x) = \frac{1}{1 - x^2}, \quad \text{a.e. } x \in I_1,$$

As for the function  $\kappa_1(x) = (1 - x^2)^{-1}$ , we have the estimate

$$0 \leq \mathbf{T}_\beta \kappa_1(x) \leq \beta \kappa_1(x) = \frac{\beta}{1 - x^2}, \quad x \in I_1.$$

*Proof.* In view of (3.4.2), we have that

$$(3.9.2) \quad \begin{aligned} \mathbf{T}_\beta \kappa_\alpha(x) &= \sum_{j \in \mathbb{Z}^\times} \frac{\beta}{(x+2j)^2} \frac{\alpha}{\alpha^2 - [s_j(x)]^2} \\ &= \sum_{j \in \mathbb{Z}^\times} \frac{\beta}{(x+2j)^2} \frac{\alpha}{\alpha^2 - \frac{\beta^2}{(x+2j)^2}} = \sum_{j \in \mathbb{Z}^\times} \frac{\alpha\beta}{\alpha^2(x+2j)^2 - \beta^2}, \end{aligned}$$

where the series converges absolutely unless it happens that a term is undefined (as the result of division by 0). Since  $s_j(x) \in I_\beta$  for  $x \in I_1$ , we see that each term is positive for  $\alpha = \beta$ , and hence

$$\mathbf{T}_\beta \kappa_\beta(x) = \mathbf{T}_\beta |\kappa_\beta|(x) = \sum_{j \in \mathbb{Z}^\times} \frac{1}{(x+2j)^2 - 1} = \frac{1}{2} \sum_{j \in \mathbb{Z}^\times} \left\{ \frac{1}{x+2j-1} - \frac{1}{x+2j+1} \right\} = \frac{1}{1-x^2},$$

by telescoping sums, as claimed. Next, since for  $0 < \beta \leq 1$  and  $j \in \mathbb{Z}^\times$ ,

$$0 \leq \frac{\beta}{(x+2j)^2 - \beta^2} \leq \frac{\beta}{(x+2j)^2 - 1}, \quad x \in I_1,$$

it follows that, by the same calculation,

$$0 \leq \mathbf{T}_\beta \kappa_1(x) \leq \sum_{j \in \mathbb{Z}^*} \frac{\beta}{(x+2j)^2 - 1} = \frac{\beta}{1-x^2}, \quad x \in I_1,$$

as claimed. The proof is complete.  $\square$

*Remark 3.9.3.* In particular, for  $\beta = 1$ , we have equality:  $\mathbf{T}_1 \kappa_1 = \kappa_1$ .

We also obtain a uniform estimate of  $\mathbf{T}_\beta^n \kappa_1$  for  $0 < \beta < 1$  and  $n = 1, 2, 3, \dots$

**Proposition 3.9.4.** Fix  $0 < \beta < 1$ . For  $n = 1, 2, 3, \dots$ , we have that

$$\mathbf{T}_\beta^n \kappa_1(x) \leq \frac{2\beta^n}{1-\beta}, \quad x \in I_1.$$

*Proof.* We first establish the asserted estimate for  $n = 1$ . As the function  $\kappa_1(x) = (1-x^2)^{-1}$  is positive, even, and convex, Proposition 3.5.1 tells us that

$$(3.9.3) \quad \mathbf{T}_\beta \kappa_1(x) \leq \beta C_1 \kappa_1\left(\frac{1}{2}\beta\right) + \frac{\beta}{(2-|x|)^2} \kappa_1\left(\frac{\beta}{2-|x|}\right) \leq \beta C_1 \kappa_1\left(\frac{1}{2}\right) + \beta \kappa_1(\beta) \leq \frac{2\beta}{1-\beta}.$$

Here, we used that  $\kappa_1$  is increasing on  $I_1^+ = ]0, 1[$ , and that  $C_1 \kappa_1(\frac{1}{2}) = \frac{4}{3}(\frac{\pi^2}{6} - 1) \leq 1$ .

Next, by iteration of Lemma 3.9.2, using that  $\mathbf{T}_\beta$  is positive, we obtain that  $\mathbf{T}_\beta^{n-1} \kappa_1 \leq \beta^{n-1} \kappa_1$ , so that a single application of the estimate (3.9.3) gives that

$$\mathbf{T}_\beta^n \kappa_1(x) = \mathbf{T}_\beta \mathbf{T}_\beta^{n-1} \kappa_1(x) \leq \beta^{n-1} \mathbf{T}_\beta \kappa_1(x) \leq \frac{2\beta^n}{1-\beta}, \quad x \in I_1,$$

as claimed.  $\square$

**3.10. The associated transfer operators.** For  $0 < \beta \leq 1$  and a function  $f \in L^1(I_1)$ , extended to vanish on  $\mathbb{R} \setminus I_1$ , we let  $\mathcal{T}_\beta f$  denote the function defined by

$$(3.10.1) \quad \mathcal{T}_\beta f(x) := \begin{cases} \sum_{j \in \mathbb{Z}} \frac{\beta}{(x+2j)^2} f\left(-\frac{\beta}{x+2j}\right), & x \in I_1, \\ 0, & x \in \mathbb{R} \setminus I_1, \end{cases}$$

whenever the sum converges absolutely. Analogously, for  $0 < \gamma \leq 1$  and a function  $f \in L^1(I_1^+)$ , extended to vanish on  $\mathbb{R} \setminus I_1^+$ , we let  $\mathcal{S}_\gamma f$  denote the function defined by

$$(3.10.2) \quad \mathcal{S}_\gamma f(x) := \begin{cases} \sum_{j=0}^{+\infty} \frac{\gamma}{(x+j)^2} f\left(\frac{\gamma}{x+j}\right), & x \in I_1^+, \\ 0, & x \in \mathbb{R} \setminus I_1^+, \end{cases}$$

whenever the sum converges absolutely. If we compare the definition of  $\mathcal{T}_\beta f$  with that of  $\mathbf{T}_\beta f$ , and the definition of  $\mathcal{S}_\gamma f$  with that of  $\mathbf{S}_\gamma f$ , we note that the index  $j = 0$  is included in the summation this time. The operators  $\mathcal{T}_\beta, \mathcal{S}_\gamma$  are *transfer operators*.

**Proposition 3.10.1.** Fix  $0 < \beta \leq 1$ . The operator  $\mathcal{T}_\beta$  is norm contractive  $L^1(I_1) \rightarrow L^1(I_1)$ . Indeed, we have that

$$\int_{-1}^1 |\mathcal{T}_\beta f(x)| dx \leq \int_{-1}^1 |f(x)| dx, \quad f \in L^1(I_1),$$

with equality if  $f \geq 0$ .

*Proof.* As a matter of definition, the function  $\mathcal{T}_\beta f$  vanishes off  $I_1$ . Next, by the triangle inequality and the change-of-variables formula, we have that

$$\begin{aligned} \int_{-1}^1 |\mathcal{T}_\beta f(x)| dx &\leq \sum_{j \in \mathbb{Z}} \int_{-1}^1 \left| f\left(-\frac{\beta}{x+2j}\right) \right| \frac{\beta dx}{(x+2j)^2} \\ &= \int_{I_1 \setminus \bar{I}_\beta} |f(t)| dt + \sum_{j \in \mathbb{Z}^\times} \int_{-\beta/(2j-1)}^{-\beta/(2j+1)} |f(t)| dt = \int_{-1}^1 |f(t)| dt, \end{aligned}$$

for  $f \in L^1(I_1)$ , understood to vanish off  $I_1$ . For  $f \geq 0$ , there is no loss in the triangle inequality, and we obtain equality.  $\square$

**Proposition 3.10.2.** *Fix  $0 < \gamma \leq 1$ . The operator  $\mathcal{S}_\gamma$  is norm contractive  $L^1(I_1^+) \rightarrow L^1(I_1^+)$ . Indeed, we have that*

$$\int_0^1 |\mathcal{S}_\gamma f(x)| dx \leq \int_0^1 |f(x)| dx, \quad f \in L^1(I_1^+),$$

with equality if  $f \geq 0$ .

The proof is analogous to that of Proposition 3.10.1 and therefore suppressed.

**3.11. Aspects from the dynamics of the Gauss-type maps.** We recall the interval notation from Subsection 3.1. For  $0 < \beta, \gamma < 1$ , the transformations  $\tau_\beta(x) = \{-\beta/x\}_2$  and  $\sigma_\gamma(x) = \{\gamma/x\}_1$  are rather degenerate on the sets  $I_1 \setminus \bar{I}_\beta$  and  $I_1^+ \setminus \bar{I}_\gamma^+$ . Indeed, the set  $I_1 \setminus \bar{I}_\beta$  is *invariant* for  $\tau_\beta$ , as  $\tau_\beta(I_1 \setminus \bar{I}_\beta) = I_1 \setminus \bar{I}_\beta$ , and the points in  $I_1 \setminus \bar{I}_\beta$  are 2-periodic, since

$$\tau_\beta \circ \tau_\beta(x) = \tau_\beta(\tau_\beta(x)) = x, \quad x \in I_1 \setminus \bar{I}_\beta.$$

In the same vein, the set  $I_1^+ \setminus \bar{I}_\gamma^+$  is invariant for  $\sigma_\gamma$ , and all points are 2-periodic, since

$$\sigma_\gamma \circ \sigma_\gamma(x) = \sigma_\gamma(\sigma_\gamma(x)) = x, \quad x \in I_1^+ \setminus \bar{I}_\gamma^+.$$

Clearly, the set  $I_1 \setminus \bar{I}_\beta$  acts as an attractor for the transformation  $\tau_\beta$ , and similarly, the set  $I_1^+ \setminus \bar{I}_\gamma^+$  acts as an attractor for the transformation  $\sigma_\gamma$ . We would like to analyze the sets of points which remain outside the attractor in a given number of steps. To this end, we put, for  $N = 2, 3, 4, \dots$ ,

$$(3.11.1) \quad \begin{aligned} \mathcal{E}_{\beta,N} &:= \{x \in \bar{I}_\beta : \tau_\beta^n(x) \in \bar{I}_\beta \text{ for } n = 1, \dots, N-1\}, \\ \mathcal{F}_{\gamma,N} &:= \{x \in \bar{I}_\gamma^+ : \sigma_\gamma^n(x) \in \bar{I}_\gamma^+ \text{ for } n = 1, \dots, N-1\}. \end{aligned}$$

where  $\tau_\beta^n := \tau_\beta \circ \dots \circ \tau_\beta$  and  $\sigma_\gamma^n := \sigma_\gamma \circ \dots \circ \sigma_\gamma$  ( $n$ -fold composition). We also agree that  $\mathcal{E}_{\beta,1} := \bar{I}_\beta$  and that  $\mathcal{F}_{\gamma,1} := \bar{I}_\gamma^+$ . The sets  $\mathcal{E}_{\beta,N}$  and  $\mathcal{F}_{\gamma,N}$  get smaller as  $N$  increases, and we form their intersections

$$(3.11.2) \quad \mathcal{E}_{\beta,\infty} := \bigcap_{N=1}^{+\infty} \mathcal{E}_{\beta,N}, \quad \mathcal{F}_{\gamma,\infty} := \bigcap_{N=1}^{+\infty} \mathcal{F}_{\gamma,N},$$

which are known as *wandering sets*, and consist of points whose orbits stay away from the attractor.

**Proposition 3.11.1.** *( $0 < \beta, \gamma < 1$ ) For  $N = 1, 2, 3, \dots$ , we have the estimates*

$$\int_{\mathcal{F}_{\gamma,N}} \frac{dt}{1+t} \leq \left( \frac{2\gamma}{1+\gamma} \right)^N \log 2 \quad \text{and} \quad \int_{\mathcal{E}_{\beta,N}} \frac{dt}{1-t^2} \leq \frac{4\beta^N}{1-\beta}.$$

As a consequence, the one-dimensional Lebesgue measures of the sets  $\mathcal{E}_{\beta,\infty}$  and  $\mathcal{F}_{\gamma,\infty}$  both vanish.

*Proof.* By inspection of the definition of the Koopman operators (3.4.1), we see that a.e. on the respective interval,

$$\mathbf{L}_\beta^N 1 = 1_{\mathcal{E}_{\beta,N}}, \quad \mathbf{K}_\gamma^N 1 = 1_{\mathcal{F}_{\gamma,N}},$$

where 1 stands for the corresponding constant function. In view of the duality (3.4.3), it follows that

$$\int_{\mathcal{F}_{\beta,N}} \frac{dt}{1+t} = \langle \lambda_1, \mathbf{K}_\gamma^N 1 \rangle_{I_1^+} = \langle \mathbf{S}_\gamma^N \lambda_1, 1 \rangle_{I_1^+} \leq \left( \frac{2\gamma}{1+\gamma} \right)^N \langle \lambda_1, 1 \rangle_{I_1^+} = \left( \frac{2\gamma}{1+\gamma} \right)^N \log 2$$

where  $\lambda_1(x) = (1+x)^{-1}$  and the estimate comes from Proposition 3.9.1. It remains to obtain the corresponding estimate for the set  $\mathcal{E}_{\beta,N}$ . Let  $\psi := 1_{I_\eta} \kappa_1$  for some  $\eta$ ,  $0 < \eta < 1$ , where  $\kappa_1(x) = (1-x^2)^{-1}$ . Then  $\psi \in L^1(I_1)$ , and we obtain from the duality (3.4.3) together with Proposition 3.9.4 that

$$\int_{I_\eta \cap \mathcal{E}_{\beta,N}} \frac{dt}{1-t^2} = \langle \psi, \mathbf{L}_\beta^N 1 \rangle_{I_1} = \langle \mathbf{T}_\beta^N \psi, 1 \rangle_{I_1} \leq \langle \mathbf{T}_\beta^N \kappa_1, 1 \rangle_{I_1} \leq \frac{2\beta^N}{1-\beta} \langle 1, 1 \rangle_{I_1} = \frac{4\beta^N}{1-\beta}.$$

Letting  $\eta \rightarrow 1$ , the remaining assertion follows by e.g. monotone convergence.

As for the sets  $\mathcal{E}_{\beta,\infty}$  and  $\mathcal{F}_{\gamma,\infty}$ , we just need to observe that right-hand sides converge to 0 geometrically, since  $2\gamma/(1+\gamma) < 1$ .  $\square$

The 2-periodicity of the points in the attractor of  $\tau_\beta$  gets reflected in the fact that the functions supported on the attractor are two-periodic for the transfer operator  $\mathcal{T}_\beta$ . Naturally, the same is true in the context of  $\sigma_\gamma$  and  $\mathcal{S}_\gamma$ . We suppress the easy proof.

**Proposition 3.11.2.** *Fix  $0 < \beta, \gamma \leq 1$ . The operator  $\mathcal{T}_\beta$  maps  $L^1(I_1 \setminus I_\beta)$  contractively into itself. Likewise,  $\mathcal{S}_\gamma$  maps  $L^1(I_1^+ \setminus I_\gamma^+)$  contractively into itself. Moreover,  $\mathcal{T}_\beta^2 f = f$  for  $f \in L^1(I_1 \setminus I_\beta)$ , and, analogously,  $\mathcal{S}_\gamma^2 f = f$  for  $f \in L^1(I_1 \setminus I_\gamma)$ .*

We shall need the following result, which describes the interlacing of the iterates of  $\mathcal{T}_\beta$  with the multiplication by characteristic functions.

**Proposition 3.11.3.** *Fix  $0 < \beta \leq 1$ . For  $N = 1, 2, 3, \dots$  and  $f \in L^1(I_1)$ , we have the identities a.e. on  $I_1$ :*

$$1_{I_\beta} \mathcal{T}_\beta^{N-1} f = \mathcal{T}_\beta^{N-1} (1_{\mathcal{E}_{\beta,N}} f), \quad \mathcal{T}_\beta^N (1_{\mathcal{E}_{\beta,N}} f) = \mathbf{T}_\beta^N f.$$

*Proof.* To simplify the presentation, we replace the  $L^1(I_1)$  function by a Dirac point mass  $f = \delta_\xi$  at an arbitrary point  $\xi \in I_1$ . If we can show that the claimed equalities holds for  $f = \delta_\xi$ , i.e.,

$$1_{I_\beta} \mathcal{T}_\beta^{N-1} \delta_\xi = \mathcal{T}_\beta^{N-1} (1_{\mathcal{E}_{\beta,N}} \delta_\xi), \quad \mathcal{T}_\beta^N (1_{\mathcal{E}_{\beta,N}} \delta_\xi) = \mathbf{T}_\beta^N \delta_\xi,$$

for Lebesgue almost every point  $\xi \in I_1$ , then the claimed equalities hold for every  $f \in L^1(I_1)$  by “averaging”. Indeed, a general  $f \in L^1(I_1)$  may be written as

$$(3.11.3) \quad f(x) = \int_{I_1} \delta_x(t) f(t) dt = \int_{I_1} \delta_t(x) f(t) dt, \quad x \in I_1,$$

where the integral is to be understood in the sense distribution theory, so that, e.g.,

$$\mathcal{T}_\beta f(x) = \int_{I_1} \mathcal{T}_\beta \delta_t(x) f(t) dt, \quad x \in I_1.$$

We first focus on the claimed identity

$$(3.11.4) \quad 1_{I_\beta} \mathcal{T}_\beta^{N-1} \delta_\xi = \mathcal{T}_\beta^{N-1} (1_{\mathcal{E}_{\beta,N}} \delta_\xi).$$

Here, we should remark that the multiplication of a point mass and a characteristic function need only make sense for almost every  $\xi \in I_1$ . For  $N = 1$ , (3.11.4) holds trivially. In the following, we consider integers  $N > 1$ . The canonical extension of the transfer operator  $\mathcal{T}_\beta$  to such point masses reads

$$(3.11.5) \quad \mathcal{T}_\beta \delta_\xi = \delta_{\tau_\beta(\xi)} = \delta_{[-\beta/\xi]_2}.$$

Note that by iteration of (3.11.5), we have

$$(3.11.6) \quad \mathcal{T}_\beta^{N-1} \delta_\xi = \delta_{\tau_\beta^{N-1}(\xi)} \quad \text{for } \xi \in I_1.$$

As a matter of definition, we know that  $\tau_\beta^{N-1}(\xi) \in \bar{I}_\beta$  for  $\xi \in \mathcal{E}_{\beta,N}$ , while for a.e.  $\xi \in I_1 \setminus \mathcal{E}_{\beta,N}$ , there exists an  $n = 1, \dots, N-1$  such that  $\tau_\beta^n(\xi) \in I_1 \setminus \bar{I}_\beta$ . As  $J_\beta = I_1 \setminus \bar{I}_\beta$  is an attractor for  $\tau_\beta$ , we conclude that for a.e.  $\xi \in I_1 \setminus \mathcal{E}_{\beta,N}$ , we have that  $\tau_\beta^{N-1}(\xi) \in I_1 \setminus \bar{I}_\beta$ . The asserted identity (3.11.4) now follows from these observations.

We turn to the remaining assertion, which claims that

$$(3.11.7) \quad \mathcal{T}_\beta^N(1_{\mathcal{E}_{\beta,N}}f) = \mathbf{T}_\beta^N f, \quad N = 1, 2, 3, \dots$$

By inspection of the definition (3.4.2) of the subtransfer operator, the action of  $\mathbf{T}_\beta$  lifts to a point mass at  $\xi \in I_1$  for a.e.  $\xi$  in the following fashion:

$$\mathbf{T}_\beta \delta_\xi = \begin{cases} \delta_{\tau_\beta(\xi)} & \text{if } \xi \in \bar{I}_\beta, \\ 0 & \text{if } \xi \in I_1 \setminus \bar{I}_\beta, \end{cases}$$

so that by iteration, again for a.e.  $\xi \in I_1$ , we obtain that

$$\mathbf{T}_\beta^N \delta_\xi = \begin{cases} \delta_{\tau_\beta^N(\xi)} & \text{if } \xi \in \mathcal{E}_{\beta,N}, \\ 0 & \text{if } \xi \in I_1 \setminus \mathcal{E}_{\beta,N}. \end{cases}$$

A comparison with the corresponding formula (3.11.6) shows that the identity (3.11.7) holds. The proof is now complete.  $\square$

The corresponding relations for  $\mathbf{S}_\gamma$  and  $\mathbf{S}_\gamma$  read as follows.

**Proposition 3.11.4.** *Fix  $0 < \gamma < 1$ . For  $N = 1, 2, 3, \dots$  and  $f \in L^1(I_1^+)$ , we have the following identities a.e. on  $I_1^+$ :*

$$1_{I_1^+} \mathbf{S}_\gamma^{N-1} f = \mathbf{S}_\beta^{N-1}(1_{\mathcal{F}_{\gamma,N}} f), \quad \mathbf{S}_\beta^N(1_{\mathcal{F}_{\gamma,N}} f) = \mathbf{S}_\gamma^N f.$$

The proof is similar to that of Proposition 3.11.3 and therefore suppressed.

**3.12. Exact endomorphisms.** We need the concept of exactness. Here, we follow the abstract approach to this notion (see e.g. [22]) and say that  $\tau_1$  (and the transfer operator  $\mathbf{T}_1$  as well) is *exact* if, in the a.e. sense,

$$\bigcap_{n=1}^{+\infty} \mathbf{L}_1^n L^\infty(I_1) = \{\text{constants}\}.$$

For  $0 < \beta < 1$ , however,  $\tau_\beta$  has a nontrivial attractor, and the notion needs to be modified. So, for  $0 < \beta < 1$ , we say that  $\tau_\beta$  (and the transfer operator  $\mathbf{T}_\beta$  as well) is *subexact* if, in the a.e. sense,

$$\bigcap_{n=1}^{+\infty} \mathbf{L}_\beta^n L^\infty(I_1) = \{0\}.$$

Mutatis mutandis, if we replace the triple  $\mathbf{T}_\beta, \mathbf{L}_\beta, I_1$  by  $\mathbf{S}_\gamma, \mathbf{K}_\gamma, I_1^+$ , we also obtain the definition of exactness and subexactness for  $\mathbf{S}_\gamma$  (and the transformation  $\sigma_\gamma$  as well).

**Proposition 3.12.1.** *Fix  $0 < \beta, \gamma < 1$ . The operators  $\mathbf{T}_\beta : L^1(I_1) \rightarrow L^1(I_1)$  and  $\mathbf{S}_\gamma : L^1(I_1^+) \rightarrow L^1(I_1^+)$  are subexact in the sense that*

$$\bigcap_{n=1}^{+\infty} \mathbf{L}_\beta^n L^\infty(I_1) = \{0\}, \quad \bigcap_{n=1}^{+\infty} \mathbf{K}_\gamma^n L^\infty(I_1^+) = \{0\}.$$

*Proof.* By inspection of the compressed Koopman operator  $\mathbf{L}_\beta^n$ , an element of the intersection

$$\bigcap_{n=1}^{+\infty} \mathbf{L}_\beta^n L^\infty(I_1)$$

is a function in  $L^\infty(I_1)$  which vanishes off the wandering set  $\mathcal{E}_{\beta,\infty}$ , but by Proposition 3.11.1, this is a null set, so the function vanishes a.e. The analogous argument applies in the case of  $\mathbf{K}_\gamma$ .  $\square$



Exactness in the case  $\beta = \gamma = 1$  is known and can be derived from the work of Thaler [32], see also Aaronson's book [2]:

**Proposition 3.12.2.** *Fix  $\beta = \gamma = 1$ . The operators  $\mathbf{T}_1 : L^1(I_1) \rightarrow L^1(I_1)$  and  $\mathbf{S}_1 : L^1(I_1^+) \rightarrow L^1(I_1^+)$  are exact in the sense that*

$$\bigcap_{n=1}^{+\infty} \mathbf{L}_1^n L^\infty(I_1) = \{\text{constants}\}, \quad \bigcap_{n=1}^{+\infty} \mathbf{K}_1^n L^\infty(I_1^+) = \{\text{constants}\}.$$

*Proof.* The map  $\tau_1$  meets the conditions (1)–(4) of Thaler's paper [32], p. 69, so by the Theorem 1 [32], p. 73,  $\mathbf{T}_1$  is exact (or, in more standard terminology,  $\tau_1$  is exact). Let us check the conditions one by one, mutatis mutandis, as he uses the interval  $[0, 1]$  and not  $I_1 = [-1, 1]$  as we do.

CONDITION (1). The fundamental intervals are given by  $B(j) := ]\frac{1}{2j+1}, \frac{1}{2j-1}[$  for  $j \in \mathbb{Z}^\times = \mathbb{Z} \setminus \{0\}$  except when  $j = \pm 1$ , when we adjoin an end point:  $B(-1) = [-1, -\frac{1}{3}[$  and  $B(1) = ]\frac{1}{3}, 1]$ . The transformation  $\tau_1$  is of  $C^2$ -class on each fundamental interval  $B(j)$ , with  $j \in \mathbb{Z}^\times$ , and has complete branches (it is "filling" in the terminology of [7]). Moreover, each fundamental interval  $B(j)$  contains exactly one fixed point  $x_j$ , and  $\tau_1'(x_j) > 1$  except on two fundamental intervals,  $B(-1)$  and  $B(1)$ , where the fixed points are the boundary points 1 and  $-1$ . On each fundamental interval  $B(j)$  we replace  $\tau_1(x) = \{-1/x\}_2$  by the appropriate branch  $\tau_{1,j}(x) = 2j - 1/x$  (this makes a difference only at the end points). The derivative at the remaining fixed points is then  $\tau_{1,-1}'(-1) = \tau_{1,1}'(1) = 1$ .

CONDITION (2). This condition is satisfied since  $\tau_1'(x) = x^{-2} \geq (1 - \epsilon)^{-2} > 1$  holds on the interval  $I_{1-\epsilon}$  within each fundamental interval  $B(j)$ .

CONDITION (3). The derivative  $\tau_1'(x) = x^{-2}$  is decreasing on  $] \frac{1}{3}, 1[$  and increasing on  $] -1, -\frac{1}{3}[$ . The remaining requirements are void.

CONDITION (4). In each fundamental interval  $B(j)$ , the expression  $|\tau_1''(x)|/\tau_1'(x)^2 = 2|x|$  is uniformly bounded.

We conclude from the definition of exactness in [32] that up to null sets,  $\{\emptyset, I_1\}$  are the only measurable subsets of  $I_1$  which for each  $n = 1, 2, 3, \dots$  may be written in the form  $\tau_1^{-n}(E_n)$  for some measurable set  $E_n \subset I_1$ . This is equivalent to having

$$\bigcap_{n=1}^{+\infty} \mathbf{L}_1^n L^\infty(I_1) = \{\text{constants}\}.$$

We turn to the Gauss map  $\sigma_1(x) = \{1/x\}_2$ , whose exactness is well-known. But we may derive it from Theorem 1 in [32] as well. However, the condition (2) is not fulfilled, as  $\sigma_1'(x) = -x^{-2} \leq -1$  in the interior of the fundamental intervals. But the iterate  $\sigma_1 \circ \sigma_1$  is uniformly expanding with  $\inf(\sigma_1 \circ \sigma_1)' > 1$ , and the conditions (1)–(4) may be verified for it. So the exactness of  $\sigma_1 \circ \sigma_1$  follows in the same fashion; this leads to

$$\bigcap_{n=1}^{+\infty} \mathbf{K}_1^{2n} L^\infty(I_1^+) = \{\text{constants}\},$$

as required.  $\square$

*Remark 3.12.3.* Some aspects of the work of Thaler [32] have been further developed by Melbourne and Terhesiu [24].

**3.13. Asymptotical behavior of the orbits of  $\mathbf{T}_\beta$  and  $\mathbf{S}_\gamma$ .** We now apply the obtained exactness to show how the iterates of  $\mathbf{T}_\beta$  and  $\mathbf{S}_\gamma$  behave.

**Proposition 3.13.1.** *Fix  $0 < \beta, \gamma < 1$ .*

(a) *For  $f \in L^1(I_1^+)$ , we have that  $\|\mathbf{S}_\gamma^n f\|_{L^1(I_1^+)} \rightarrow 0$  as  $n \rightarrow +\infty$ .*

(b) *For  $f \in L^1(I_1)$ , we have that  $\|\mathbf{T}_\beta^n f\|_{L^1(I_1)} \rightarrow 0$  as  $n \rightarrow +\infty$ .*

*Proof.* This follows from Proposition 3.12.1 combined with Theorem 4.3 in [22].  $\square$

**Proposition 3.13.2.** Fix  $\beta = \gamma = 1$ .

(a) For  $f \in L^1(I_1^+)$  with  $\langle f, 1 \rangle_{I_1^+} = 0$ , we have that  $\|\mathbf{S}_1^n f\|_{L^1(I_1^+)} \rightarrow 0$  as  $n \rightarrow +\infty$ .

(b) For  $f \in L^1(I_1)$  with  $\langle f, 1 \rangle_{I_1} = 0$ , we have that  $\|\mathbf{T}_1^n f\|_{L^1(I_1)} \rightarrow 0$  as  $n \rightarrow +\infty$ .

*Proof.* This follows from Proposition 3.12.2 combined with Theorem 4.3 in [22].  $\square$

There is a weak analogue of Proposition 3.13.1(b) which applies for  $\beta = 1$ . The proof is based on the fact that the absolutely continuous invariant measure has infinite mass.

**Proposition 3.13.3.** Fix  $\beta = 1$ . For  $f \in L^1(I_1)$ , we have that for fixed  $\eta$ ,  $0 < \eta < 1$ ,

$$\lim_{n \rightarrow +\infty} \int_{-\eta}^{\eta} |\mathbf{T}_1^n f(x)| dx = 0.$$

*Proof.* Since pointwise  $|\mathbf{T}_1^n f| \leq \mathbf{T}_1^n |f|$ , we may assume without loss of generality that  $f \geq 0$ . We recall the notation  $\kappa_1(x) = (1 - x^2)^{-1}$ , and pick a number  $\xi$  with  $0 < \xi < 1$ . Let  $g$  be the function

$$g(x) := \frac{\langle f, 1 \rangle_{I_1}}{\langle 1_{I_\xi} \kappa_1, 1 \rangle_{I_1}} 1_{I_\xi}(x) \kappa_1(x), \quad x \in I_1.$$

Then  $g \in L^1(I_1)$ , and

$$\langle f - g, 1 \rangle_{I_1} = 0.$$

By Proposition 3.13.2(b), we conclude that  $\|\mathbf{T}_1^n(f - g)\|_{L^1(I_1)} \rightarrow 0$  as  $n \rightarrow +\infty$ . Moreover, by the triangle inequality, we have that

$$\|\mathbf{T}_1^n f\|_{L^1(I_\eta)} \leq \|\mathbf{T}_1^n(f - g)\|_{L^1(I_1)} + \|\mathbf{T}_1^n g\|_{L^1(I_\eta)}.$$

Since the function  $g$  is positive, and

$$g(x) \leq \frac{\langle f, 1 \rangle_{I_1}}{\langle 1_{I_\xi} \kappa_1, 1 \rangle_{I_1}} \kappa_1(x),$$

we see that

$$(3.13.1) \quad \|\mathbf{T}_1^n g\|_{L^1(I_\eta)} = \langle \mathbf{T}_1^n g, 1_{I_\eta} \rangle_{I_1} \leq \frac{\langle f, 1 \rangle_{I_1}}{\langle 1_{I_\xi} \kappa_1, 1 \rangle_{I_1}} \langle \mathbf{T}_1^n \kappa_1, 1_{I_\eta} \rangle_{I_1} = \frac{\langle f, 1 \rangle_{I_1}}{\langle 1_{I_\xi} \kappa_1, 1 \rangle_{I_1}} \langle \kappa_1, 1_{I_\eta} \rangle_{I_1},$$

since  $\mathbf{T}_1 \kappa_1 = \kappa_1$  (see Lemma 3.9.2). Moreover, since

$$\langle 1_{I_\xi} \kappa_1, 1 \rangle_{I_1} \rightarrow +\infty \quad \text{as } \xi \rightarrow 1,$$

we may get the norm  $\|\mathbf{T}_1^n g\|_{L^1(I_\eta)}$  as small as we like for fixed  $\eta$  by letting  $\xi$  be appropriately close to 1. This means that the right hand side of (3.13.1) may be as close to 0 as we want, the first term by letting  $n$  be big, and the second, by letting  $\xi$  be close to 1. The proof is complete.  $\square$

#### 4. BACKGROUND MATERIAL: THE HARDY AND BMO SPACES ON THE LINE

**4.1. The Hardy  $H^1$ -space; analytic and real.** For a reference on the basic facts of Hardy spaces and BMO (bounded mean oscillation), we refer to, e.g., the monographs of Duren and Garnett [9], [13], as well as those of Stein [29], [30], and Stein and Weiss [31].

Let  $H_+^1(\mathbb{R})$  and  $H_-^1(\mathbb{R})$  be the subspaces of  $L^1(\mathbb{R})$  consisting of those functions whose Poisson extensions to the upper half plane

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

are holomorphic and conjugate-holomorphic, respectively. Here, we use the term conjugate-holomorphic (or anti-holomorphic) to mean that the complex conjugate of the function in question is holomorphic.

It is well-known that any function  $f \in H_+^1(\mathbb{R})$  has vanishing integral,

$$(4.1.1) \quad \langle f, 1 \rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(t) dt = 0, \quad f \in H_+^1(\mathbb{R}).$$

In other words,  $H_+^1(\mathbb{C}) \subset L_0^1(\mathbb{R})$ , where

$$(4.1.2) \quad L_0^1(\mathbb{R}) := \{f \in L^1(\mathbb{R}) : \langle f, 1 \rangle_{\mathbb{R}} = 0\}.$$

In fact, there is a related Fourier analysis characterization of the Hardy space  $H_+^1(\mathbb{R})$  and  $H_-^1(\mathbb{R})$ : for  $f \in L^1(\mathbb{R})$ ,

$$(4.1.3) \quad f \in H_+^1(\mathbb{R}) \iff \forall y \geq 0 : \int_{\mathbb{R}} e^{iyt} f(t) dt = 0$$

and

$$(4.1.4) \quad f \in H_-^1(\mathbb{R}) \iff \forall y \leq 0 : \int_{\mathbb{R}} e^{iyt} f(t) dt = 0.$$

We will refer to the space

$$H_{\otimes}^1(\mathbb{R}) := H_+^1(\mathbb{R}) \oplus H_-^1(\mathbb{R})$$

as the *real  $H^1$ -space of the line  $\mathbb{R}$* . Here, the symbol  $\oplus$  means the direct sum, i.e, the elements of  $f \in H_{\otimes}^1(\mathbb{R})$  are functions  $f \in L_0^1(\mathbb{R})$  which may be written in the form

$$(4.1.5) \quad f = f_1 + f_2, \quad \text{where } f_1 \in H_+^1(\mathbb{R}), f_2 \in H_-^1(\mathbb{R}),$$

plus the fact that  $H_+^1(\mathbb{R}) \cap H_-^1(\mathbb{R}) = \{0\}$ , which is a Fourier-analytic consequence of (4.1.3) and (4.1.4). Obviously, we have the inclusion  $H_{\otimes}^1(\mathbb{R}) \subset L_0^1(\mathbb{R})$ ; it is perhaps slightly less obvious that  $H_{\otimes}^1(\mathbb{R})$  is dense in  $L_0^1(\mathbb{R})$  in the norm of  $L^1(\mathbb{R})$ . It is clear that the decomposition (4.1.5) is unique. As for notation, we let  $\mathbf{P}_+$  and  $\mathbf{P}_-$  denote the projections  $\mathbf{P}_+ f := f_1$  and  $\mathbf{P}_- f := f_2$  in the decomposition (4.1.5). These Szegő projections  $\mathbf{P}_+, \mathbf{P}_-$  can of course be extended beyond this  $H_{\otimes}^1(\mathbb{R})$  setting; more about this in the following subsection.

**4.2. The BMO space and the modified Hilbert transform.** With respect to the dual action

$$\langle f, g \rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(t)g(t)dt,$$

we may identify the dual space of  $H_{\otimes}^1(\mathbb{R})$  with  $\text{BMO}(\mathbb{R})/\mathbb{C}$ . Here,  $\text{BMO}(\mathbb{R})$  is the space of functions of *bounded mean oscillation*; this is the celebrated *Fefferman duality theorem* [10], [11]. As for notation, we write “ $\cdot/\mathbb{C}$ ” to express that we mod out with respect to the constant functions. One of the main results in the theory is the theorem of Fefferman and Stein [11] which tells us that

$$(4.2.1) \quad \text{BMO}(\mathbb{R}) = L^\infty(\mathbb{R}) + \tilde{\mathbf{H}}L^\infty(\mathbb{R}).$$

or, in words, a function  $g$  is in  $\text{BMO}(\mathbb{R})$  if and only if it may be written in the form  $g = g_1 + \tilde{\mathbf{H}}g_2$ , where  $g_1, g_2 \in L^\infty(\mathbb{R})$ . Here,  $\tilde{\mathbf{H}}$  denotes the *modified Hilbert transform*, defined for  $f \in L^\infty(\mathbb{R})$  by the formula

$$(4.2.2) \quad \tilde{\mathbf{H}}f(x) := \frac{1}{\pi} \text{pv} \int_{\mathbb{R}} f(t) \left\{ \frac{1}{x-t} + \frac{t}{1+t^2} \right\} dt = \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R} \setminus [x-\epsilon, x+\epsilon]} f(t) \left\{ \frac{1}{x-t} + \frac{t}{1+t^2} \right\} dt.$$

The decomposition (4.2.1) is clearly not unique. The non-uniqueness of the decomposition is equal to the intersection space

$$(4.2.3) \quad H_{\otimes}^\infty(\mathbb{R}) := L^\infty(\mathbb{R}) \cap \tilde{\mathbf{H}}L^\infty(\mathbb{R}),$$

the *real  $H^\infty$ -space*.

We should compare the modified Hilbert transform  $\tilde{\mathbf{H}}$  with the standard *Hilbert transform*  $\mathbf{H}$ , which acts boundedly on  $L^p(\mathbb{R})$  for  $1 < p < +\infty$ , and maps  $L^1(\mathbb{R})$  into  $L^{1,\infty}(\mathbb{R})$  for  $p = 1$ . Here,  $L^{1,\infty}(\mathbb{R})$  denotes the *weak  $L^1$ -space*, see Subsection 7.1 below. The Hilbert transform of a function  $f$ , assumed integrable on the line  $\mathbb{R}$  with respect to the measure  $(1+t^2)^{-1/2}dt$ , is defined as the principal value integral

$$(4.2.4) \quad \mathbf{H}f(x) := \frac{1}{\pi} \text{pv} \int_{\mathbb{R}} f(t) \frac{dt}{x-t} = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\pi} \int_{\mathbb{R} \setminus [x-\epsilon, x+\epsilon]} f(t) \frac{dt}{x-t}.$$

If  $f \in L^p(\mathbb{R})$ , where  $1 \leq p < +\infty$ , then both  $\mathbf{H}f$  and  $\tilde{\mathbf{H}}f$  are well-defined a.e., and it is easy to see that the difference  $\tilde{\mathbf{H}}f - \mathbf{H}f$  equals to a constant. It is often useful to think of the natural harmonic extensions of the Hilbert transforms  $\mathbf{H}f$  and  $\tilde{\mathbf{H}}f$  to the upper half-plane  $\mathbb{C}_+$  given by

$$(4.2.5) \quad \mathbf{H}f(z) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{\operatorname{Re} z - t}{|z - t|^2} f(t) dt, \quad \tilde{\mathbf{H}}f(z) := \frac{1}{\pi} \int_{\mathbb{R}} \left\{ \frac{\operatorname{Re} z - t}{|z - t|^2} + \frac{t}{t^2 + 1} \right\} f(t) dt.$$

So, as a matter of normalization, we have that  $\tilde{\mathbf{H}}f(i) = 0$ . This tells us the value of the constant mentioned above:  $\tilde{\mathbf{H}}f - \mathbf{H}f = -\mathbf{H}f(i)$ .

Returning to the real  $H^1$ -space, we note the following characterization of the space in terms of the Hilbert transform: for  $f \in L^1(\mathbb{R})$ ,

$$f \in H_{\otimes}^1(\mathbb{R}) \iff f \in L_0^1(\mathbb{R}) \text{ and } \mathbf{H}f \in L_0^1(\mathbb{R});$$

see Proposition 7.1.1 later on.

The Szegő projections  $\mathbf{P}_+$  and  $\mathbf{P}_-$  which were mentioned in Subsection 4.1 are more generally defined in terms of the Hilbert transform:

$$(4.2.6) \quad \mathbf{P}_+ f := \frac{1}{2}(f + i\mathbf{H}f), \quad \mathbf{P}_- f := \frac{1}{2}(f - i\mathbf{H}f).$$

In a similar manner, for  $f \in L^\infty(\mathbb{R})$ , based on the modified Hilbert transform  $\tilde{\mathbf{H}}$  we may define the corresponding modified Szegő projections (which are actually projections modulo the constant functions)

$$(4.2.7) \quad \tilde{\mathbf{P}}_+ f := \frac{1}{2}(f + i\tilde{\mathbf{H}}f), \quad \tilde{\mathbf{P}}_- f := \frac{1}{2}(f - i\tilde{\mathbf{H}}f),$$

so that, by definition,  $f = \tilde{\mathbf{P}}_+ f + \tilde{\mathbf{P}}_- f$ . If we are given two functions  $f \in H_{\otimes}^1(\mathbb{R})$  and  $g \in L^\infty(\mathbb{R})$ , the dual action  $\langle \cdot, \cdot \rangle_{\mathbb{R}}$  naturally splits into holomorphic and conjugate-holomorphic parts:

$$(4.2.8) \quad \langle f, g \rangle_{\mathbb{R}} = \langle \mathbf{P}_+ f, \tilde{\mathbf{P}}_- g \rangle_{\mathbb{R}} + \langle \mathbf{P}_- f, \tilde{\mathbf{P}}_+ g \rangle_{\mathbb{R}}.$$

Modulo the constants, the space  $\operatorname{BMO}(\mathbb{R})$  naturally splits into holomorphic and conjugate-holomorphic components:

$$(4.2.9) \quad \operatorname{BMO}(\mathbb{R})/\mathbb{C} = [\operatorname{BMOA}^+(\mathbb{R})/\mathbb{C}] \oplus [\operatorname{BMOA}^-(\mathbb{R})/\mathbb{C}].$$

The spaces appearing on the right-hand side,  $\operatorname{BMOA}^+(\mathbb{R})$  and  $\operatorname{BMOA}^-(\mathbb{R})$ , denote the subspaces of  $\operatorname{BMO}(\mathbb{R})$  consisting of functions with Poisson extensions to the upper half-plane  $\mathbb{C}_+$  that are holomorphic and conjugate-holomorphic, respectively.

The operator  $\tilde{\mathbf{H}}$  makes sense also on functions from  $\operatorname{BMO}(\mathbb{R})$ . It is then natural to ask what is  $\tilde{\mathbf{H}}^2$ :

**Lemma 4.2.1.** *For  $f \in L^p(\mathbb{R})$ ,  $1 < p < +\infty$ , we have that  $\mathbf{H}^2 f = -f$ . Moreover, for  $f \in L^\infty(\mathbb{R})$ , we have that  $\tilde{\mathbf{H}}^2 f = -f + c(f)$ , where  $c(f)$  is the constant*

$$c(f) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{t^2 + 1} dt.$$

*Proof.* The assertion for  $1 < p < +\infty$  is completely standard (see any textbook in Harmonic Analysis). We turn to the assertion for  $p = +\infty$ . First, we observe that without loss of generality, we may assume  $f$  is real-valued. Then the function  $2\tilde{\mathbf{P}}_+ f$  is the holomorphic function in the upper half-plane whose real part is the Poisson extension of  $f$ , and the choice of the imaginary part is fixed by the requirement  $2\operatorname{Im} \tilde{\mathbf{P}}_+ f(i) = \tilde{\mathbf{H}}f(i) = 0$ . The function

$$-2i\tilde{\mathbf{P}}_+ f = \tilde{\mathbf{H}}f - if$$

extends to a holomorphic function in the upper half-plane  $\mathbb{C}_+$ , with real part  $\tilde{\mathbf{H}}f$ . So we may identify  $-f$  with  $\tilde{\mathbf{H}}^2 f$  up to an additive constant. The additive constant is determined by the requirement that  $\tilde{\mathbf{H}}^2 f(i) = 0$ , and so  $\tilde{\mathbf{H}}^2 f(i) = -f + f(i) = -f + c(f)$ . Here,  $f(i)$  is understood in terms of Poisson extension.  $\square$

**4.3. BMO and the Fourier transform.** The Fourier transform of a function  $f \in L^1(\mathbb{R})$  is given by

$$(4.3.1) \quad \hat{f}(x) := \int_{\mathbb{R}} e^{i\pi xt} f(t) dt,$$

and it is well understood how to extend the Fourier transform to the setting of tempered distributions (see, e.g., [18]). It is well-known how to characterize in terms of the Fourier transform the spaces  $\text{BMOA}^+(\mathbb{R})$  and  $\text{BMOA}^-(\mathbb{R})$  as subspaces of  $\text{BMO}(\mathbb{R})$ . We state these known facts as a lemma (without supplying a proof). We recall the notation  $\mathbb{R}_+ = [0, +\infty[$  and  $\mathbb{R}_- = ]-\infty, 0]$ .

**Lemma 4.3.1.** *Suppose  $f \in \text{BMO}(\mathbb{R})$ . Then  $f \in \text{BMOA}^+(\mathbb{R})$  if and only if  $\hat{f}$  is supported on the interval  $\mathbb{R}_-$ . Likewise,  $f \in \text{BMOA}^-(\mathbb{R})$  if and only if  $\hat{f}$  is supported on the interval  $\mathbb{R}_+$ .*

**4.4. The BMO space of 2-periodic functions.** We shall need the space

$$\text{BMO}(\mathbb{R}/2\mathbb{Z}) := \{f \in \text{BMO}(\mathbb{R}) : f(t+2) \equiv f(t)\},$$

that is, the BMO space of 2-periodic functions. Via the complex exponential mapping  $t \mapsto e^{i\pi t}$  ( $\mathbb{R} \mapsto \mathbb{T}$ ), we identify the unit circle  $\mathbb{T}$  with  $\mathbb{R}/2\mathbb{Z}$ :  $\mathbb{R}/2\mathbb{Z} \cong \mathbb{T}$ , and the space  $\text{BMO}(\mathbb{R}/2\mathbb{Z})$  is then just the standard BMO space on  $\mathbb{T}$ . Let us write

$$\text{BMOA}^+(\mathbb{R}/2\mathbb{Z}) := \text{BMOA}^+(\mathbb{R}) \cap \text{BMO}(\mathbb{R}/2\mathbb{Z})$$

and

$$\text{BMOA}^-(\mathbb{R}/2\mathbb{Z}) := \text{BMOA}^-(\mathbb{R}) \cap \text{BMO}(\mathbb{R}/2\mathbb{Z}),$$

for the subspaces of  $\text{BMO}(\mathbb{R}/2\mathbb{Z})$  that consist of functions whose Poisson extensions to the upper half-plane  $\mathbb{C}_+$  are holomorphic and conjugate-holomorphic, respectively.

As  $L^2$ -integrable functions on the “circle”  $\mathbb{R}/2\mathbb{Z}$ , the elements of the space  $\text{BMO}(\mathbb{R}/2\mathbb{Z})$  have (a.e. convergent) Fourier series expansions. This means that the Fourier transform  $\hat{f}$  of a function  $f \in \text{BMO}(\mathbb{R}/2\mathbb{Z})$ , defined by (4.3.1) and interpreted in the sense of distribution theory, is a sum of Dirac point masses along the integers  $\mathbb{Z}$ . We formalize this observation as a lemma.

**Lemma 4.4.1.** *Suppose  $f \in \text{BMO}(\mathbb{R})$ . Then  $f \in \text{BMO}(\mathbb{R}/2\mathbb{Z})$  if and only if the distribution  $\hat{f}$  is supported on the integers  $\mathbb{Z}$ , and at each point of  $\mathbb{Z}$ , it is a Dirac point mass.*

This result is well-known.

## 5. THE ZARISKI CLOSURES OF TWO PORTIONS OF THE LATTICE-CROSS

**5.1. An involution and the modified Hilbert transform on BMO.** For a positive real parameter  $\beta$ , let  $J_\beta^*$  be the involutive operator defined by

$$(5.1.1) \quad J_\beta^* f(x) := f(-\beta/x), \quad x \in \mathbb{R}^\times.$$

We recall the definition (4.2.2) of the modified Hilbert transform  $\tilde{H}$ .

**Lemma 5.1.1.** *For  $f \in \text{BMO}(\mathbb{R})$  and a positive real  $\beta$ , we have that*

$$(J_\beta^* \tilde{H} f)(x) = (\tilde{H} J_\beta^* f)(x) + c_\beta(f),$$

where  $c_\beta(f)$  is the constant

$$c_\beta(f) := \tilde{H} f(i\beta) = (\beta^2 - 1) \int_{\mathbb{R}} \frac{tf(t) dt}{(1+t^2)(\beta^2+t^2)}.$$

*Proof.* Without loss of generality, we may assume that  $f$  is real-valued. The mapping  $x \mapsto -\beta/x$  extends to a conformal automorphism of the upper half-plane given by  $z \mapsto -\beta/z$ , and the function  $2\tilde{P}_+ f$  is a holomorphic function in the upper half plane with real part equal to the Poisson extension of  $f$ . We realize that the functions  $J_\beta^* \tilde{P}_+ f$  and  $J_\beta^* \tilde{P}_+ f$  differ by an imaginary constant. Taking imaginary parts, the result follows by plugging at the point  $z = i$ .  $\square$

### 5.2. The Zariski closure of the portions of the lattice-cross on the space-like cone boundary.

Recall that  $1_E$  stands the characteristic function of the set  $E$ , which equals 1 on  $E$  and 0 off of  $E$ . The Fourier transform of the function  $e^{i/t}$  in the sense of Schwartzian distributions may be known, but we have no specific reference.

**Proposition 5.2.1.** *In the sense of distribution theory on the real line  $\mathbb{R}$  we have that,*

$$\lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}} e^{i/t + itx - \epsilon|t|} \frac{dt}{2\pi} = \delta_0(x) - 1_{\mathbb{R}_+}(x) x^{-1/2} J_1(2x^{1/2}),$$

where  $\delta_0$  is the unit Dirac point mass at 0, and  $J_1$  denotes the standard Bessel function, so that

$$x^{-1/2} J_1(2x^{1/2}) = \sum_{j=0}^{+\infty} \frac{(-1)^j}{j!(j+1)!} x^j.$$

*Proof.* A direct calculation can be obtained on the basis of formula 3.324 in [14]. A less cumbersome approach is to compute the Fourier transform of the function  $H_1(x) := 1_{\mathbb{R}_+}(x) x^{-1/2} J_1(2x^{1/2})$ :

$$\hat{H}_1(y) = \int_{\mathbb{R}} e^{i\pi xy} H_1(x) dx = \int_0^{+\infty} e^{i\pi xy} x^{-1/2} J_1(2x^{1/2}) dx = 2 \int_0^{+\infty} e^{i\pi y t^2} J_1(2t) dt,$$

where the integral is absolutely convergent for  $\text{Im } y > 0$  and has a well-defined interpretation on the real line  $\mathbb{R}$ , e.g., in terms of nontangential limits. From the standard Bessel function asymptotics, we know that

$$|H_1(x)| = O(x^{-3/4}) \quad \text{as } x \rightarrow +\infty,$$

so that, in particular,  $H_1 \in L^2(\mathbb{R})$ . By basic Hardy space theory, the nontangential limit interpretation from the upper half-plane agrees with the standard  $L^2$  Fourier transform on the line  $\mathbb{R}$ . By application of formula 6.631 in [14], we have that, for  $\text{Im } y > 0$ ,

$$\hat{H}_1(y) = 2 \int_0^{+\infty} e^{i\pi y t^2} J_1(2t) dt = e^{-i/(2\pi y)} M_{0, \frac{1}{2}} \left( \frac{i}{\pi y} \right),$$

where the function on the right-hand side is of *Whittaker type*. In view of the integral representation of such Whittaker functions (formula 9.221 in [14]) we find that

$$\hat{H}_1(y) = 1 - e^{-i/(\pi y)}, \quad \text{Im } y > 0,$$

and, in a second step, that the above identification of the Fourier transform holds in the  $L^2$ -sense a.e. on  $\mathbb{R}$ . Since the Fourier transform of the Dirac delta  $\delta_0$  is the constant function 1, the assertion of the proposition now follows from the Fourier inversion formula.  $\square$

*Proof of Theorem 1.8.1.* We obviously have the inclusions

$$\Lambda_{\alpha, \beta}^{++} \subset \text{zclos}_{\Gamma_M}(\Lambda_{\alpha, \beta}^{++}), \quad \Lambda_{\alpha, \beta}^{--} \subset \text{zclos}_{\Gamma_M}(\Lambda_{\alpha, \beta}^{--}),$$

and it remains to show that the Zariski closure contains no extraneous points. We will focus our attention to the set  $\Lambda_{\alpha, \beta}^{++}$ ; the treatment of the set  $\Lambda_{\alpha, \beta}^{--}$  is analogous. In view of (1.4.4) (which relates  $\hat{\mu}(\xi)$  to the compressed measure  $\pi_1 \mu$ ) we need to do the following. Given a point  $\xi^* = (\xi_1^*, \xi_2^*) \in \mathbb{R}^2 \setminus \Lambda_{\alpha, \beta}^{++}$ , we need to find a finite complex-valued absolutely continuous Borel measure  $\nu$  on  $\mathbb{R}^\times$ , such that

$$\int_{\mathbb{R}^\times} e^{i\pi[\xi_1^* t + M^2 \xi_2^* / (4\pi^2 t)]} d\nu(t) \neq 0,$$

while at the same time

$$\int_{\mathbb{R}^\times} e^{i\pi \alpha m t} d\nu(t) = \int_{\mathbb{R}^\times} e^{iM^2 \beta n / (4\pi t)} d\nu(t) = 0, \quad m, n \in \mathbb{Z}_{+, 0}.$$

By a scaling argument, we may without loss of generality restrict our attention to the normalized case  $\alpha := 1$  and  $M := 2\pi$ . As  $\nu$  is absolutely continuous, we may write  $d\nu(t) := g(t)dt$ , where  $g \in L^1(\mathbb{R})$ . Given the above normalization, we need  $g$  to satisfy

$$(5.2.1) \quad \int_{\mathbb{R}^\times} e^{i\pi[\xi_1^* t + \xi_2^* / t]} g(t) dt \neq 0,$$

where

$$(\xi_1^*, \xi_2^*) \in \mathbb{R}^2 \setminus [(\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_+)],$$

while at the same time

$$(5.2.2) \quad \int_{\mathbb{R}^\times} e^{i\pi m t} g(t) dt = \int_{\mathbb{R}^\times} e^{i\pi \beta n / t} g(t) dt = 0, \quad m, n \in \mathbb{Z}_{+,0}.$$

We will try to find such a function  $g$  in the slightly smaller space  $H_\otimes^1(\mathbb{R})$ . To analyze the condition (5.2.2), we might as well study the weak-star closures in the dual space  $\text{BMO}(\mathbb{R})/\mathbb{C}$  of the linear spans of (i) the functions  $t \mapsto e^{i\pi m t}$ , with  $m \in \mathbb{Z}_{+,0}$ , and of (ii) the functions  $t \mapsto e^{i\pi \beta n / t}$ , with  $n \in \mathbb{Z}_{+,0}$ . In the first case, we obtain the subspace  $\text{BMOA}^+(\mathbb{R}/2\mathbb{Z})/\mathbb{C}$  (see Subsection 4.4 for the notation). In the second case, we obtain instead the subspace  $\text{BMOA}_{(\beta)}^-(\mathbb{R})/\mathbb{C}$ , where  $\text{BMOA}_{(\beta)}^-(\mathbb{R}) = \mathbf{J}_\beta^* \text{BMOA}^-(\mathbb{R}/2\mathbb{Z})$  and the operator  $\mathbf{J}_\beta^*$  is as in (5.1.1). Now, for  $g \in H_\otimes^1(\mathbb{R})$ , (5.2.2) expresses that  $g$  annihilates the sum space  $\text{BMOA}^+(\mathbb{R}/2\mathbb{Z}) + \text{BMOA}_{(\beta)}^-(\mathbb{R})$ .

To simplify the notation, we let  $F_0 \in L^\infty(\mathbb{R})$  be the function  $F_0(t) := e^{i\pi[\xi_1^* t + \xi_2^* / t]}$ . Then, in view of (4.2.8), we have that

$$\langle g, F_0 \rangle_{\mathbb{R}} = \langle \mathbf{P}_+ g, \tilde{\mathbf{P}}_- F_0 \rangle_{\mathbb{R}} + \langle \mathbf{P}_- g, \tilde{\mathbf{P}}_+ F_0 \rangle_{\mathbb{R}}.$$

It follows that if we can obtain that

$$(5.2.3) \quad \tilde{\mathbf{P}}_+ F_0 \notin \text{BMOA}^+(\mathbb{R}/2\mathbb{Z}) \quad \text{or} \quad \tilde{\mathbf{P}}_- F_0 \notin \text{BMOA}_{(\beta)}^-(\mathbb{R}),$$

then we are done, because we are free to choose  $g \in H_\otimes^1(\mathbb{R})$  as we like. Indeed, if  $\tilde{\mathbf{P}}_+ F_0 \notin \text{BMOA}^+(\mathbb{R}/2\mathbb{Z})$ , then we just pick a  $g \in H_-^1(\mathbb{R})$  which does not annihilate  $\text{BMOA}^+(\mathbb{R}/2\mathbb{Z})$ , and if  $\tilde{\mathbf{P}}_- F_0 \notin \text{BMOA}_{(\beta)}^-(\mathbb{R})$ , then we just pick a  $g \in H_+^1(\mathbb{R})$  which does not annihilate  $\text{BMOA}_{(\beta)}^-(\mathbb{R})$ . In each case, we achieve (5.2.1). Using  $\mathbf{J}_\beta^*$ , we see by Lemma 5.1.1) that (5.2.3) is equivalent to having

$$(5.2.4) \quad \tilde{\mathbf{P}}_+ F_0 \notin \text{BMOA}^+(\mathbb{R}/2\mathbb{Z}) \quad \text{or} \quad \tilde{\mathbf{P}}_- \mathbf{J}_\beta^* F_0 \notin \text{BMOA}^-(\mathbb{R}/2\mathbb{Z}).$$

Moreover, the function  $F_1 := \mathbf{J}_\beta^* F_0$  is of the same general type as  $F_0$ :  $F_1(t) = e^{-i\pi[\eta_1^* t + \eta_2^* / t]}$ , where  $\eta_1^* := \xi_2^* / \beta$  and  $\eta_2^* := \beta \xi_1^*$ . We can bring this one step further, and consider  $F_2(t) := e^{i\pi[\eta_1^* t + \eta_2^* / t]}$  (this is just the complex conjugate of  $F_1(t)$ ), and express the requirement (5.2.4) in the form

$$(5.2.5) \quad \tilde{\mathbf{P}}_+ F_0 \notin \text{BMOA}^+(\mathbb{R}/2\mathbb{Z}) \quad \text{or} \quad \tilde{\mathbf{P}}_+ F_2 \notin \text{BMOA}^+(\mathbb{R}/2\mathbb{Z}).$$

By combining Lemmas 4.3.1 and 4.4.1 with Proposition 5.2.1 in the appropriate manner, using that the Bessel function  $J_1$  is real-analytic (so that its zero set is a discrete set of points), we find that

$$\tilde{\mathbf{P}}_+ F_0 \in \text{BMOA}^+(\mathbb{R}/2\mathbb{Z}) \iff \xi^* = (\xi_1^*, \xi_2^*) \in (\bar{\mathbb{R}}_- \times \bar{\mathbb{R}}_+) \cup (\mathbb{Z}_+ \times \{0\}).$$

The analogous case with  $F_2$  in place of  $F_0$  reads

$$\tilde{\mathbf{P}}_+ F_2 \in \text{BMOA}^+(\mathbb{R}/2\mathbb{Z}) \iff \xi^* (\xi_1^*, \xi_2^*) \in (\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_-) \cup (\{0\} \times \beta\mathbb{Z}_+).$$

As we put these assertion together, it becomes clear that

$$\tilde{\mathbf{P}}_+ F_0, \tilde{\mathbf{P}}_+ F_2 \in \text{BMOA}^+(\mathbb{R}/2\mathbb{Z}) \iff (\xi_1^*, \xi_2^*) \in (\mathbb{Z}_{+,0} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}_+).$$

The set of  $\xi^*$  in the right-hand side expression is precisely the excluded set of points on the lattice-cross, and we conclude that (5.2.5) must hold. This completes the proof of the theorem.  $\square$

## 6. DYNAMIC UNIQUE CONTINUATION FROM ONE BRANCH OF THE HYPERBOLA TO THE OTHER

**6.1. Dynamic unique continuation and the critical density case.** We recall the definition of the hyperbola  $\Gamma_M$  and its branch  $\Gamma_M^+$  from the introduction, see (1.4.2) and (1.6.1). Here, we will supply the proof of Theorem 1.6.1. As Theorem 1.6.1 is somewhat defective at the critical regime  $\alpha\beta M^2 = 16\pi^2$ , we may ask whether adding an additional point to the lattice-cross  $\Lambda_{\alpha,\beta}$  might improve the situation. Indeed, this turns out to be the case, provided that the point we add is on the cross (but not on the lattice-cross itself, of course):

**Theorem 6.1.1.** *Fix  $0 < \alpha, \beta, M < +\infty$ . Suppose  $\alpha\beta M^2 = 16\pi^2$ , and pick a point  $\xi^* \in (\mathbb{R} \times \{0\}) \times (\{0\} \times \mathbb{R})$  on the cross, which is not in  $\Lambda_{\alpha,\beta}$ . If we write  $\Lambda_{\alpha,\beta}^* := \Lambda_{\alpha,\beta} \cup \{\xi^*\}$ , then  $(\Gamma_M^+, \Lambda_{\alpha,\beta}^*)$  is a Heisenberg uniqueness pair.*

Theorem 6.1.1 has a reformulation in terms of unique continuation from  $\Gamma_M^+$  to  $\Gamma_M$ , which we think of as an example of *dynamic unique continuation*.

**Corollary 6.1.2.** *Fix  $0 < \alpha, \beta, M < +\infty$ . Suppose  $\alpha\beta M^2 = 16\pi^2$ , and pick a point  $\xi^* \in (\mathbb{R} \times \{0\}) \times (\{0\} \times \mathbb{R})$  on the cross, which is not in  $\Lambda_{\alpha,\beta}$ . If we write  $\Lambda_{\alpha,\beta}^* := \Lambda_{\alpha,\beta} \cup \{\xi^*\}$ , then any measure  $\mu \in \text{AC}(\Gamma_M, \Lambda_{\alpha,\beta}^*)$  is uniquely determined by its restriction to the hyperbola branch  $\Gamma_M^+$ .*

We first supply the proof of Theorem 1.6.1, and then proceed with the proof of Theorem 6.1.1.

*Proof of Theorem 1.6.1.* We pick an arbitrary measure  $\mu \in \text{AC}(\Gamma_M, \Lambda_{\alpha,\beta})$  and form its  $x_1$ -compression  $\nu := \pi_1 \mu$  which is a finite absolutely continuous complex measure on the positive half-axis  $\mathbb{R}_+$ . By a scaling argument, we may assume that that

$$\alpha = 2, \quad M = 2\pi.$$

Since  $\nu$  is absolutely continuous, we may write  $d\nu(t) = f(t)dt$ , where  $f \in L^1(\mathbb{R}_+)$ . We observe that the vanishing condition  $\hat{\mu} = 0$  on  $\Lambda_{\alpha,\beta}$  with  $\alpha = 2$  and  $M = 2\pi$  amounts to having

$$(6.1.1) \quad \int_{\mathbb{R}_+} e^{i2\pi mt} f(t) dt = \int_{\mathbb{R}_+} e^{i2\pi \gamma n/t} f(t) dt = 0, \quad m, n \in \mathbb{Z},$$

where  $\gamma := \beta/2$ . It was shown in [7] that for  $2 < \beta < +\infty$ , there is an infinite-dimensional space of solutions  $f$ . So, in the sequel, we will restrict the parameter  $\beta$  to  $0 < \beta \leq 2$ , and hence  $\gamma$  to  $0 < \gamma \leq 1$ . To complete the proof of the theorem, we need to show that

- (i) for  $0 < \gamma < 1$ , the condition (6.1.1) entails that  $f = 0$  holds a.e. on  $\mathbb{R}_+$ , whereas
- (ii) for  $\gamma = 1$ , (6.1.1) implies that  $f = C_0 f_0$  holds a.e. on  $\mathbb{R}_+$  for some constant  $C_0$ , where  $f_0$  is the function

$$(6.1.2) \quad f_0(t) := \frac{1_{[0,1]}(t)}{1+t} - \frac{1_{[1,+\infty]}(t)}{t(1+t)}.$$

As a first step, we rewrite (6.1.1) in the form

$$(6.1.3) \quad \int_{\mathbb{R}_+} e^{i2\pi mt} f(t) dt = \int_{\mathbb{R}_+} e^{i2\pi nt} f\left(\frac{\gamma}{t}\right) \frac{dt}{t^2} = 0, \quad m, n \in \mathbb{Z}.$$

Next, for a function  $g \in L^1(\mathbb{R}_+)$  and an integer  $m \in \mathbb{Z}$  we have that that

$$(6.1.4) \quad \begin{aligned} \int_{\mathbb{R}_+} e^{i2\pi mt} g(t) dt &= \sum_{j=0}^{+\infty} \int_{[j, j+1]} e^{i2\pi mt} g(t) dt \\ &= \sum_{j=0}^{+\infty} \int_{[0,1]} e^{i2\pi mt} g(t+j) dt = \int_{[0,1]} e^{i2\pi mt} \sum_{j=0}^{+\infty} g(t+j) dt. \end{aligned}$$



Together with the uniqueness theorem for Fourier series, (6.1.4) now shows that

$$(6.1.5) \quad \int_{\mathbb{R}_+} e^{i2\pi mt} g(t) dt = 0 \quad \forall m \in \mathbb{Z} \iff \sum_{j=0}^{+\infty} g(t+j) = 0 \text{ a.e. on } \mathbb{R}_+.$$

As we apply (6.1.5) to the two cases  $g(t) = f(t)$  and  $g(t) = t^{-2}f(\gamma/t)$ , the conditions of (6.1.3) find an equivalent formulation:

$$(6.1.6) \quad \sum_{j=0}^{+\infty} f(t+j) = \sum_{j=0}^{+\infty} \frac{1}{(t+j)^2} f\left(\frac{\gamma}{t+j}\right) = 0 \text{ a.e. on } \mathbb{R}_+.$$

We single out the first term in each sum, and rewrite (6.1.6) further:

$$(6.1.7) \quad f(t) = - \sum_{j=1}^{+\infty} f(t+j), \quad \frac{1}{t^2} f\left(\frac{\gamma}{t}\right) = - \sum_{j=1}^{+\infty} \frac{1}{(t+j)^2} f\left(\frac{\gamma}{t+j}\right),$$

in both cases a.e. on  $\mathbb{R}_+$ . After the change-of-variables  $t \mapsto \gamma/t$  in the second condition, (6.1.7) becomes

$$(6.1.8) \quad f(t) = - \sum_{j=1}^{+\infty} f(t+j), \quad f(t) = - \sum_{j=1}^{+\infty} \frac{\gamma^2}{(\gamma+jt)^2} f\left(\frac{\gamma t}{\gamma+jt}\right),$$

again a.e. on  $\mathbb{R}_+$ . By combining the conditions of equality in (6.1.8), we find that

$$(6.1.9) \quad f(t) = \sum_{j,l=1}^{+\infty} \frac{\gamma^2}{[\gamma+l(j+t)]^2} f\left(\frac{\gamma(t+j)}{\gamma+l(t+j)}\right), \text{ a.e. on } \mathbb{R}_+.$$

Now, it is easy to check that after restriction to the interval  $I_1^+ = ]0, 1[$ , condition (6.1.9) amounts to having

$$(6.1.10) \quad f = \mathbf{S}_\gamma^2 f \text{ a.e. on } I_1^+,$$

where  $\mathbf{S}_\gamma$  is the subtransfer operator as given by (3.4.2). If  $0 < \gamma < 1$ , Proposition 3.13.1(a) tells us that  $\mathbf{S}_\gamma^{2n} f \rightarrow 0$  in  $L^1(I_1^+)$  as  $n \rightarrow +\infty$ , so the only way the equality (6.1.10) is possible is if  $f = 0$  a.e. on  $I_1^+$ . But then the second equality in (6.1.8) gives that  $f = 0$  a.e. on  $\mathbb{R} \setminus I_1^+$ , and hence  $f = 0$  a.e. on  $\mathbb{R}_+$ , as desired. This settles (i).

We turn to the remaining case  $\gamma = 1$ . It is well-known that the function  $\lambda_1(t) = (1+t)^{-1}$  is an invariant density on  $I_1^+$  for the Gauss map  $\theta_1(t) = \{1/t\}_1$  (cf. Subsection 3.9). In terms of the transfer operator  $\mathbf{S}_1$ , this means that  $\mathbf{S}_1 \lambda_1 = \lambda_1$ , so that  $\mathbf{S}_1^2 \lambda_1 = \lambda_1$  as well. Next, we consider the function

$$h := f - \frac{\langle 1, f \rangle_{I_1^+}}{\log 2} \lambda_1 \in L^1(I_1^+),$$

which by construction has  $\langle h, 1 \rangle_{I_1^+} = 0$  and  $h = \mathbf{S}_1^2 h$ . By iteration, the latter property entails that  $h$  has  $h = \mathbf{S}_1^{2n} h$  for  $n = 1, 2, 3, \dots$ , so that in view of Proposition 3.13.2(a), we have that

$$h = \mathbf{S}_1^{2n} h \rightarrow 0 \text{ as } n \rightarrow +\infty,$$

where the convergence is in the norm of  $L^1(I_1^+)$ , which implies that  $h = 0$  a.e. on  $I_1^+$ . It is now immediate that

$$f = C_0 \lambda_1 \text{ a.e. on } I_1^+, \text{ where } C_0 := \frac{\langle 1, f \rangle_{I_1^+}}{\log 2} \in \mathbb{C}.$$

Next, the second identity in (6.1.8) with  $\gamma = 1$  tells us what  $f_0$  equals on the remaining set  $\mathbb{R}_+ \setminus I_1^+$ :

$$\begin{aligned} f(t) &= -C_0 \sum_{j=1}^{+\infty} \frac{1}{(1+jt)^2} \lambda_1\left(\frac{t}{1+jt}\right) = -C_0 \sum_{j=1}^{+\infty} \frac{1}{(1+jt)^2} \frac{1}{1+\frac{t}{1+jt}} \\ &= -C_0 \sum_{j=1}^{+\infty} \frac{1}{(1+jt)(1+(j+1)t)} = -\frac{C_0}{t} \sum_{j=1}^{+\infty} \left\{ \frac{1}{1+jt} - \frac{1}{1+(j+1)t} \right\} = -\frac{C_0}{t(1+t)}. \end{aligned}$$

The conclusion that  $f = C_0 f_0$  a.e. on  $\mathbb{R}_+$  is now immediate, where  $f_0$  is given by (6.1.2) and  $C_0 \in \mathbb{C}$  is a constant. Finally, it is an exercise to verify that the function  $f_0$  indeed satisfies (6.1.6), so that  $f_0$  (and its complex constant multiples) meets the vanishing condition for the Fourier transform, as expressed in (6.1.1). This settles (ii), and the the proof is complete.  $\square$

*Proof of Theorem 6.1.1.* As before, rescaling allows us to fix the parameter values:

$$\alpha = \beta = 2, \quad M = 2\pi,$$

which corresponds to  $\gamma = \beta/2 = 1$  in the preceding proof. We need to show that if  $\mu \in \text{AC}(\Gamma_M, \Lambda_{\alpha,\beta}^*)$ , then  $\mu = 0$  as a measure. Since  $\Lambda_{\alpha,\beta}^* \supset \Lambda_{\alpha,\beta}$ , and we are in the critical parameter regime in terms of Theorem 1.6.1, we have that necessarily  $d\pi_1\mu(t) = C_0 f_0(t)$ , where  $f_0$  is given by (6.1.2), and  $C_0$  is a complex constant. We recall that  $\Lambda_{\alpha,\beta}^* = \Lambda_{\alpha,\beta}^* \cup \{\xi^*\}$ , for some point  $\xi^* = (\xi_1^*, \xi_2^*)$  with either  $\xi_1^* = 0$  or  $\xi_2^* = 0$ , which is not the lattice-cross  $\Lambda_{\alpha,\beta}$ . By symmetry, both cases are equivalent, and we choose to consider  $\xi_2^* = 0$ , so that  $\xi^* = (\xi_1^*, 0)$ , where  $\xi_1^* \in \mathbb{R} \setminus \alpha\mathbb{Z} = \mathbb{R} \setminus 2\mathbb{Z}$ . The Fourier transform of  $\mu$  restricted to the axis  $\mathbb{R} \times \{0\}$  equals (cf. (1.4.4))

$$\begin{aligned} (6.1.11) \quad \hat{\mu}(\xi_1, 0) &= \int_{\mathbb{R}^\times} e^{i\pi\xi_1 t} d\pi_1\mu(t) = C_0 \int_{\mathbb{R}^\times} e^{i\pi\xi_1 t} f_0(t) dt \\ &= C_0 \left\{ \int_{[0,1]} e^{i\pi\xi_1 t} \frac{dt}{1+t} - \int_{[1,+\infty[} e^{i\pi\xi_1 t} \frac{dt}{t(1+t)} \right\} \\ &= C_0 \left\{ \int_{[0,1]} e^{i\pi\xi_1 t} \frac{dt}{1+t} - \int_{[1,+\infty[} e^{i\pi\xi_1 t} \left( \frac{1}{t} - \frac{1}{1+t} \right) dt \right\} \\ &= C_0 \left\{ \int_{[0,+\infty[} e^{i\pi\xi_1 t} \frac{dt}{1+t} - \int_{[1,+\infty[} e^{i\pi\xi_1 t} \frac{dt}{t} \right\} = C_0 (e^{-i\pi\xi_1} - 1) \int_{[1,+\infty[} e^{i\pi\xi_1 t} \frac{dt}{t}. \end{aligned}$$

Here, in the rightmost expression, the integral should be understood as a generalized Riemann integral. Since our additional vanishing condition is  $\hat{\mu}(\xi_1^*, 0) = 0$ , above calculation (6.1.11) tells us that this is the same as

$$C_0 (e^{-i\pi\xi_1^*} - 1) \int_{[1,+\infty[} e^{i\pi\xi_1^* t} \frac{dt}{t} = 0.$$

Moreover, since  $\xi_1^*$  is real but not an even integer, we know that  $e^{i\pi\xi_1^*} \neq 1$ , and the above equation simplifies to

$$(6.1.12) \quad C_0 \int_{[1,+\infty[} e^{i\pi\xi_1^* t} \frac{dt}{t} = 0.$$

Splitting the above generalized Riemann integral into real and imaginary parts, we see that

$$\int_1^{+\infty} e^{i\pi\xi_1^* t} \frac{dt}{t} = \int_1^{+\infty} \cos(\pi\xi_1^* t) \frac{dt}{t} + i \int_1^{+\infty} \sin(\pi\xi_1^* t) \frac{dt}{t}.$$

The real and imaginary parts may be expressed in terms the rather standard functions “si” and “ci”:

$$\int_1^{+\infty} \cos(\pi\xi_1^* t) \frac{dt}{t} = \int_{\pi|\xi_1^*|}^{+\infty} \frac{\cos y}{y} dy = -\text{ci}(\pi|\xi_1^*|),$$

and

$$\int_1^{+\infty} \sin(\pi \xi_1^* t) \frac{dt}{t} = \operatorname{sgn}(\xi_1^*) \int_{\pi|\xi_1^*|}^{+\infty} \frac{\sin y}{y} dy = -\operatorname{sgn}(\xi_1^*) \operatorname{si}(\pi|\xi_1^*|),$$

so that

$$\int_1^{+\infty} e^{i\pi \xi_1^* t} \frac{dt}{t} = -\operatorname{ci}(\pi|\xi_1^*|) - i \operatorname{sgn}(\xi_1^*) \operatorname{si}(\pi|\xi_1^*|).$$

Here, we write  $\operatorname{sgn}(x) = x/|x|$  for the standard sign function. We now observe that is rather well-known that the parametrization

$$\operatorname{ci}(\pi x) + i \operatorname{si}(\pi x), \quad 0 < x < +\infty,$$

forms the *Nielsen* (or *sici*) spiral which converges to the origin as  $x \rightarrow +\infty$ , and whose curvature is proportional to  $x$  (see, e.g. [1]). In particular, the spiral never intersects the origin, which tells us that

$$\int_{[1,+\infty[} e^{i\pi \xi_1^* t} \frac{dt}{t} \neq 0,$$

and, therefore, (6.1.12) gives us that  $C_0 = 0$  and consequently that  $\mu = 0$  as a measure. The proof is complete.  $\square$

## 7. THE HILBERT TRANSFORM ON $L^1$ AND THE PREDUAL OF REAL $H^\infty$ ON THE LINE

**7.1. The Hilbert transform on  $L^1$ .** For background material on the Hilbert transform and related topics, see, e.g. the monographs [9], [13], [29], [30], and [31].

Let  $L^{1,\infty}(\mathbb{R})$  denote the *weak  $L^1$ -space*, i.e., the space of Lebesgue measurable functions  $f : \mathbb{R} \rightarrow \mathbb{C}$  such that the set

$$E_f(\lambda) := \{x \in \mathbb{R} : |f(x)| > \lambda\}, \quad \lambda \in \mathbb{R}_+,$$

enjoys the estimate (the absolute value of a measurable subset of  $\mathbb{R}$  stands for its Lebesgue measure)

$$|E_f(\lambda)| \leq \frac{C_f}{\lambda}, \quad \lambda \in \mathbb{R}_+;$$

the optimal constant  $C_f$  is written  $\|f\|_{L^{1,\infty}(\mathbb{R})}$ ; it is the  $L^{1,\infty}(\mathbb{R})$ -*quasinorm* of  $f$ . By identifying functions that coincide almost everywhere, the space  $L^{1,\infty}(\mathbb{R})$  is a *quasi-Banach space*. It is well-known that the Hilbert transform as given by (4.2.4) maps  $\mathbf{H} : L^1(\mathbb{R}) \rightarrow L^{1,\infty}(\mathbb{R})$ . Note, however, that functions in  $L^{1,\infty}(\mathbb{R})$  are rather wild and, e.g., it is not immediately clear how to associate such a function with a distribution. However, there is another interpretation of the Hilbert transform as a mapping from  $L^1(\mathbb{R})$  into a space of distributions on  $\mathbb{R}$ , and it is good to know that these interpretations of  $\mathbf{H}f$  for a given  $f \in L^1(\mathbb{R})$  are in a one-to-one correspondence. The weak  $L^1$ -space associated with an interval  $I$  (or a set of positive Lebesgue measure), written  $L^{1,\infty}(I)$ , is defined analogously.

If for the moment we use the symbol  $\mathbf{F}$  to denote the Fourier transform, then the Hilbert transform is  $\mathbf{H} = -i\mathbf{F}^{-1}\mathbf{M}_{\operatorname{sgn}}\mathbf{F}$ , where  $\mathbf{M}_{\operatorname{sgn}}$  stands for multiplication by the sign function  $\operatorname{sgn}$ . Thus, after taking the Fourier transform, the distributional interpretation of the Hilbert is that of multiplication by the unimodular function which takes the value  $-i$  on the positive half-line, and the value  $i$  on the negative half-line. The distributional interpretation can also be implemented more directly:

$$(7.1.1) \quad \langle \varphi, \mathbf{H}f \rangle_{\mathbb{R}} := -\langle \mathbf{H}\varphi, f \rangle_{\mathbb{R}},$$

where  $\varphi$  is a test function with compact support, and  $f \in L^1(\mathbb{R})$ . Note that  $\mathbf{H}\varphi$ , the Hilbert transform of the test function, may be defined without the need of the principal value integral:

$$\mathbf{H}\varphi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\varphi(x-t) - \varphi(x+t)}{t} dt;$$

it is a  $C^\infty$  function on  $\mathbb{R}$  with decay  $\mathbf{H}\varphi(x) = O(|x|^{-1})$  as  $|x| \rightarrow +\infty$ . As a consequence, it is clear from (7.1.1) how to extend the notion  $\mathbf{H}f$  to functions  $f$  with  $(|x|+1)^{-1}f(x)$  in  $L^1(\mathbb{R})$ .

Our next proposition characterizes the space  $H_{\otimes}^1(\mathbb{R})$ . For the proof, we need the notation for the open unit disk:

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

**Proposition 7.1.1.** *Suppose  $f \in L^1(\mathbb{R})$ . Then the following are equivalent:*

- (i)  $f \in H_{\otimes}^1(\mathbb{R})$ .
- (ii)  $\mathbf{H}f \in L^1(\mathbb{R})$ , where  $\mathbf{H}f$  is understood as a distribution on the line  $\mathbb{R}$ .
- (iii)  $\mathbf{H}f \in L^1(\mathbb{R})$ , where  $\mathbf{H}f$  is understood as an almost everywhere defined function in  $L^{1,\infty}(\mathbb{R})$ .

*Proof.* The implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) are trivial, so we turn to the remaining implication (iii) $\Rightarrow$ (i). This result, however, is the real line analogue of the result for the circle in [20], p. 87. The transfer to the unit disk is handled by an appropriate Moebius map from  $\mathbb{D}$  to  $\mathbb{C}_+$ .  $\square$

A first application of Proposition 7.1.1 gives us the following result.

**Corollary 7.1.2.** *Suppose  $f \in L^1(\mathbb{R})$ , and that  $\mathbf{H}f = 0$  pointwise almost everywhere on  $\mathbb{R}$ . Then  $f = 0$  almost everywhere.*

*Proof.* Without loss of generality,  $f$  is real-valued. In view of Proposition 7.1.1,  $f \in H_{\otimes}^1(\mathbb{R})$ , and as a consequence, the function  $F := f + i\mathbf{H}f$  is in  $H_+^1(\mathbb{R})$ . But on the real line,  $F$  is real-valued, so that the Poisson extension of  $F$  to  $\mathbb{C}_+$  is real-valued as well. But this Poisson extension is holomorphic in  $\mathbb{C}_+$ , so  $F$  must be constant, and the constant is seen to be 0.  $\square$

*Remark 7.1.3.* We note that there are the closely related theories of reflectionless measures (see, e.g., [25]) and of real outer functions [12].

**7.2. The real  $H^\infty$  space.** The real  $H^\infty$  space is denoted by  $H_{\otimes}^\infty(\mathbb{R})$ , and it consists of all functions  $f \in L^\infty(\mathbb{R})$  of the form

$$(7.2.1) \quad f = f_1 + f_2, \quad f_1 \in H_+^\infty(\mathbb{R}), \quad f_2 \in H_-^\infty(\mathbb{R}).$$

Here,  $H_+^\infty(\mathbb{R})$  consists of all functions in  $L^\infty(\mathbb{R})$  whose Poisson extension to the upper half-plane is holomorphic, while  $H_-^\infty(\mathbb{R})$  consists of all functions in  $L^\infty(\mathbb{R})$  whose Poisson extension to the upper half-plane is conjugate-holomorphic (alternatively, the Poisson extension to the lower half-plane is holomorphic). The decomposition (7.2.1) is unique up to additive constants. Equipped with the natural norm,  $H_{\otimes}^\infty(\mathbb{R})$  is a Banach space.

The content of next proposition is well-known. For the convenience of the reader, we supply the simple proof.

**Proposition 7.2.1.** *We have the equivalence*

$$f \in H_{\otimes}^\infty(\mathbb{R}) \iff f, \tilde{\mathbf{H}}f \in L^\infty(\mathbb{R}).$$

*Proof.* If  $f \in H_{\otimes}^\infty(\mathbb{R})$ , then  $f = f_1 + f_2$ , where  $f_1 \in H_+^\infty(\mathbb{R})$  and  $f_2 \in H_-^\infty(\mathbb{R})$ . Since  $\tilde{\mathbf{H}}f = i(f_2 - f_1) + c$ , where  $c$  is the constant that makes  $\tilde{\mathbf{H}}f(i) = 0$ , we see that  $\tilde{\mathbf{H}}f \in L^\infty(\mathbb{R})$ .

On the other hand, if  $f, \tilde{\mathbf{H}}f \in L^\infty(\mathbb{R})$ , then  $f + i\tilde{\mathbf{H}}f \in H_+^\infty(\mathbb{R})$  and  $f - i\tilde{\mathbf{H}}f \in H_-^\infty(\mathbb{R})$ , so that

$$2f = (f + i\tilde{\mathbf{H}}f) + (f - i\tilde{\mathbf{H}}f) \in H_{\otimes}^\infty(\mathbb{R}).$$

The proof is complete.  $\square$

**7.3. The predual of real  $H^\infty$ .** We shall be concerned with the following space of distributions on the line  $\mathbb{R}$ :

$$\mathfrak{L}(\mathbb{R}) := L^1(\mathbb{R}) + \mathbf{H}L_0^1(\mathbb{R}),$$

which we supply with the appropriate norm

$$(7.3.1) \quad \|u\|_{\mathfrak{L}(\mathbb{R})} := \inf \left\{ \|f\|_{L^1(\mathbb{R})} + \|g\|_{L^1(\mathbb{R})} : u = f + \mathbf{H}g, \quad f \in L^1(\mathbb{R}), \quad g \in L_0^1(\mathbb{R}) \right\},$$

which makes  $\mathfrak{L}(\mathbb{R})$  a Banach space.

We recall that  $L_0^1(\mathbb{R})$  is the codimension-one subspace of  $L^1(\mathbb{R})$  which consists of the functions whose integral over  $\mathbb{R}$  vanishes. Given  $f \in L^1(\mathbb{R})$  and  $g \in L_0^1(\mathbb{R})$ , the action of  $u := f + \mathbf{H}g$  on a test function  $\varphi$  is (compare with (7.1.1))

$$(7.3.2) \quad \langle \varphi, f + \mathbf{H}g \rangle_{\mathbb{R}} = \langle \varphi, f \rangle_{\mathbb{R}} - \langle \mathbf{H}\varphi, g \rangle_{\mathbb{R}} = \langle \varphi, f \rangle_{\mathbb{R}} - \langle \tilde{\mathbf{H}}\varphi, g \rangle_{\mathbb{R}};$$

we observe that the last identity uses that  $\langle 1, g \rangle_{\mathbb{R}} = 0$  and the fact that the functions  $\tilde{\mathbf{H}}\varphi$  and  $\mathbf{H}\varphi$  differ by a constant.

**OBSERVATION.** In view of Proposition 7.2.1, the right hand side of (7.3.2) makes sense for  $\varphi \in H_{\otimes}^{\infty}(\mathbb{R})$ . To be more precise, in accordance with (7.3.2), every  $\varphi \in H_{\otimes}^{\infty}(\mathbb{R})$  defines a continuous linear functional on  $\mathfrak{V}(\mathbb{R})$ .

It remains to identify the dual space of  $\mathfrak{V}(\mathbb{R})$  with  $H_{\otimes}^{\infty}(\mathbb{R})$ .

**Proposition 7.3.1.** *Each continuous linear functional  $\mathfrak{V}(\mathbb{R}) \rightarrow \mathbb{C}$  corresponds to a function  $\varphi \in H_{\otimes}^{\infty}(\mathbb{R})$  in accordance with (7.3.2). In short, the dual space of  $\mathfrak{V}(\mathbb{R})$  equals  $H_{\otimes}^{\infty}(\mathbb{R})$ .*

*Proof.* A standard approximation argument involving test functions can be used to establish that  $L^1(\mathbb{R})$  is a dense subspace of  $\mathfrak{V}(\mathbb{R})$ . As the inclusion map  $L^1(\mathbb{R}) \rightarrow \mathfrak{V}(\mathbb{R})$  is continuous, it follows that every continuous linear functional  $\mathfrak{V}(\mathbb{R}) \rightarrow \mathbb{C}$  restricts to a continuous linear functional  $L^1(\mathbb{R})$ , which by standard functional analysis corresponds to an element  $\varphi \in L^{\infty}(\mathbb{R})$ . By density and continuity,  $\varphi$  determines the linear functional completely. As  $\varphi \in L^{\infty}(\mathbb{R})$ , we see that  $\tilde{\mathbf{H}}\varphi \in \text{BMO}(\mathbb{R})$ . By (7.3.2),  $\tilde{\mathbf{H}}\varphi$  must give a continuous linear functional  $L_0^1(\mathbb{R}) \rightarrow \mathbb{C}$ . It is easy to see that this is only possible if  $\tilde{\mathbf{H}}\varphi \in L^{\infty}(\mathbb{R})$ , which completes the proof, by Proposition 7.2.1.  $\square$

The space  $\mathfrak{V}(\mathbb{R})$  is a Banach space, and Proposition 7.3.1 asserts that its dual space is  $H_{\otimes}^{\infty}(\mathbb{R})$  (the real  $H^{\infty}$  space). For this reason, we will refer to  $\mathfrak{V}(\mathbb{R})$  as the (canonical) *predual of real  $H^{\infty}$* .

**Remark 7.3.2.** Since an  $L^1$ -function  $f$  gives rise to an absolutely continuous measure  $f(t)dt$ , it is natural to think of  $\mathfrak{V}(\mathbb{R})$  as embedded into the space  $\mathfrak{M}(\mathbb{R}) := M(\mathbb{R}) + \mathbf{H}M_0(\mathbb{R})$ , where  $M(\mathbb{R})$  denotes the space of complex-valued finite Borel measures on  $\mathbb{R}$ , and  $M_0(\mathbb{R})$  is the subspace of measures  $\mu \in M(\mathbb{R})$  with  $\mu(\mathbb{R}) = 0$ . The Hilbert transforms of singular measures noticeably differ from those of absolutely continuous measures (see [26]).

**7.4. The “valeur au point” function associated with an element of the predual of real  $H^{\infty}$ .** We recall that  $\mathfrak{V}(\mathbb{R})$  consists of distributions on the real line. However, the definition

$$\mathfrak{V}(\mathbb{R}) = L^1(\mathbb{R}) + \mathbf{H}L_0^1(\mathbb{R})$$

would allow us to also think of this space as a subspace of  $L^{1,\infty}(\mathbb{R})$ , the weak  $L^1$ -space. It is a natural question to wonder about the relationship between the distribution and the  $L^{1,\infty}$  function. We will stick to the distribution theory definition of  $\mathfrak{V}(\mathbb{R})$ , and associate with a given  $u \in \mathfrak{V}(\mathbb{R})$  the “valeur au point” function  $\text{vp}[u]$  at almost all points of the line. The precise definition of  $\text{vp}[u]$  is as follows.

**Definition 7.4.1.** For a fixed  $x \in \mathbb{R}$ , let  $\chi = \chi_x$  is a compactly supported  $C^{\infty}$ -smooth function on  $\mathbb{R}$  with  $\chi(t) = 1$  for all  $t$  in an open neighborhood of the point  $x$ . Also, let

$$P_{x+i\epsilon}(t) := \pi^{-1} \frac{\epsilon}{\epsilon^2 + (x-t)^2}$$

be the Poisson kernel. The *valeur au point function* associated with the distribution  $u$  on  $\mathbb{R}$  is the function  $\text{vp}[u] = \text{vp}[u\chi]$  given by

$$(7.4.1) \quad \text{vp}[u](x) := \lim_{\epsilon \rightarrow 0^+} \langle \chi P_{x+i\epsilon}, u \rangle_{\mathbb{R}}, \quad x \in \mathbb{R},$$

wherever the limit exists.

In principle,  $\text{vp}[u](x)$  might depend on the choice of the cut-off function  $\chi$ . The following lemma guarantees that this is not the case in the relevant situation.

**Lemma 7.4.2.** *For  $u = f + \mathbf{H}g \in \mathfrak{L}(\mathbb{R})$ , where  $f \in L^1(\mathbb{R})$  and  $g \in L_0^1(\mathbb{R})$ , the valeur au point function  $\text{vp}[u](x)$  does not depend on the choice of the cut-off  $\chi$ . Moreover, we have that*

$$\text{vp}[u](x) = f(x) + \mathbf{H}g(x), \quad \text{a.e. } x \in \mathbb{R},$$

where on the right hand side, the function  $\mathbf{H}g(x)$  is defined pointwise as a principal value.

*Proof.* For  $f \in L^1(\mathbb{R})$ , it is a standard exercise involving Poisson integrals to show that  $\text{vp}[f](x) = f(x)$  holds for almost all  $x \in \mathbb{R}$  (for details, see, e.g., [13], Chapter 1), and that the choice of  $\chi$  does not matter for the value of  $\text{vp}[f](x)$  for a given point  $x \in \mathbb{R}$ .

We turn to the evaluation of  $\text{vp}[\mathbf{H}g](x)$ . By translation invariance, we may as well consider only  $x = 0$ . As a matter of definition, we have that

$$\begin{aligned} (7.4.2) \quad \text{vp}[\mathbf{H}g](0) &= \lim_{\epsilon \rightarrow 0^+} \langle \chi P_{i\epsilon}, \mathbf{H}g \rangle_{\mathbb{R}} = - \lim_{\epsilon \rightarrow 0^+} \langle \mathbf{H}[\chi P_{i\epsilon}], g \rangle_{\mathbb{R}} \\ &= \lim_{\epsilon \rightarrow 0^+} \left\{ \langle \mathbf{H}[\tilde{\chi} P_{i\epsilon}], g \rangle_{\mathbb{R}} - \langle \mathbf{H}[P_{i\epsilon}], g \rangle_{\mathbb{R}} \right\}, \end{aligned}$$

where  $\tilde{\chi} := 1 - \chi$  and  $\chi$  is a smooth cut-off function with  $\chi(t) = 1$  near  $t = 0$ . Here, as above,  $P_{i\epsilon}$  is the function

$$P_{i\epsilon}(t) = \pi^{-1} \frac{\epsilon}{\epsilon^2 + t^2},$$

and its Hilbert transform is given by

$$\mathbf{H}[P_{i\epsilon}](t) = \pi^{-1} \frac{t}{\epsilon^2 + t^2}.$$

A calculation reveals that

$$\pi^{-1} \frac{t}{\epsilon^2 + t^2} = \int_0^{+\infty} \frac{1_{\mathbb{R} \setminus [-\tau, \tau]} }{\pi t} \frac{2\epsilon^2 \tau}{(\epsilon^2 + \tau^2)^2} d\tau,$$

which can be used to show that

$$- \lim_{\epsilon \rightarrow 0^+} \langle \mathbf{H}[P_{i\epsilon}](t), g \rangle_{\mathbb{R}} = - \lim_{\tau \rightarrow 0^+} \int_{\mathbb{R} \setminus [-\tau, \tau]} \frac{g(t)}{\pi t} dt = \mathbf{H}g(0),$$

where the rightmost equality sign is a matter of the pointwise definition of the Hilbert transform. The desired conclusion now follows from (7.4.2), once we have established that for fixed  $\tilde{\chi}$ , we have

$$\|\mathbf{H}[\tilde{\chi} P_{i\epsilon}]\|_{L^\infty(\mathbb{R})} = O(\epsilon)$$

as  $\epsilon \rightarrow 0^+$ . This is rather elementary and left to the interested reader; here, we only observe that the function  $\tilde{\chi}$  is smooth and bounded, which equals 1 near infinity and vanishes near the origin, so that  $\tilde{\chi} P_{i\epsilon}$  becomes a very small and quite smooth function.  $\square$

Additional properties of the mapping  $\text{vp}$  are outlined below.

**Proposition 7.4.3.** (Kolmogorov) *The mapping  $\text{vp} : \mathfrak{L}(\mathbb{R}) \rightarrow L^{1,\infty}(\mathbb{R})$ ,  $u \mapsto \text{vp}[u]$ , is continuous.*

*Proof.* This follows from the standard weak-type estimate for the Hilbert transform (see, e.g., [13]).  $\square$

The next result allows us to identify  $u$  with  $\text{vp}[u]$ .

**Proposition 7.4.4.** (Kolmogorov) *If  $u \in \mathfrak{L}(\mathbb{R})$  and  $\text{vp}[u] = 0$  almost everywhere on  $\mathbb{R}$ , then  $u = 0$  as a distribution.*

*Proof.* We write  $u = f + \mathbf{H}g$ , where  $f \in L^1(\mathbb{R})$  and  $g \in L_0^1(\mathbb{R})$ . Since  $g \in L_0^1(\mathbb{R})$  and, by assumption,  $\text{vp}[g] = -f \in L^1(\mathbb{R})$ , it follows from Proposition 7.1.1 that  $g \in H_{\odot}^1(\mathbb{R})$  and consequently that  $\mathbf{H}g \in L^1(\mathbb{R})$  as a distribution. Since the Hilbert transform  $\mathbf{H}$  leaves the space  $H_{\odot}^1(\mathbb{R})$  invariant, we also obtain that  $f \in H_{\odot}^1(\mathbb{R})$ , and then it is immediate from the assumption that  $u = 0$  as a distribution.  $\square$

The local version of Proposition 7.4.4 runs as follows.

**Proposition 7.4.5.** *If  $u \in \mathfrak{L}(\mathbb{R})$  and  $\text{vp}[u] = 0$  almost everywhere on an open interval  $I \subset \mathbb{R}$ , then the distribution  $u$  is supported on  $\mathbb{R} \setminus I$ .*

*Proof.* We split  $u = f + \mathbf{H}g$ , where  $f \in L^1(\mathbb{R})$  and  $g \in L_0^1(\mathbb{R})$ . Without loss of generality, we may assume that  $f$  and  $g$  are real-valued. Again, without loss of generality, the open interval  $I$  is assumed to be *bounded*. By the classical theorem of Kolmogorov [9], the function  $G := g + i\mathbf{H}g$  is in the  $H^p$ -space in the upper half plane (with respect to Poissonian measure  $\pi^{-1}(1+t^2)^{-1}dt$  on the real line), for each  $p$  with  $0 < p < 1$ . In Kolmogorov's theorem,  $\mathbf{H}g$  initially has the pointwise interpretation, but in a second step, it is valid with the distributional interpretation as well. By assumption,  $\text{vp}[\mathbf{H}g] = -f$  holds on the bounded open interval  $I$ , so that the boundary function for  $G$  is in  $L^1$  on  $I$ . Essentially, this means that  $G$  is in  $H^1$  near  $I$  in the upper half-plane. This can be made precise in the following manner. We choose a slightly smaller interval  $J \subset I$ , whose both endpoints differ from those of  $I$ . Next, we choose a bounded simply connected Jordan domain  $\Omega$  in the upper half-plane  $\mathbb{C}_+$  whose boundary curve  $\partial\Omega$  is  $C^\infty$ -smooth, with the property that  $\partial\Omega \cap \mathbb{R} = J$ . Then it is not difficult to see that  $G$ , restricted to  $\Omega$ , belongs to the  $H^1$ -space on  $\Omega$ , which is most conveniently defined in terms of a fixed conformal mapping from the unit disk  $\mathbb{D}$  onto  $\Omega$ . The remaining part of the proof is an exercise in Schwarzian reflection across the interval  $J$ .  $\square$

**7.5. Dual action on intervals.** If  $I \subset \mathbb{R}$  is an open interval, and  $f, g : I \rightarrow \mathbb{C}$  are two Borel measurable functions with  $fg \in L^1(I)$ , then we may define *the dual action on  $I$* :

$$\langle f, g \rangle_I := \int_I f(t)g(t)dt;$$

this is a special case of dual action on a more general measurable set (see Subsection 3.2). For instance, if  $f$  is a test function with compact support in  $I$ , and  $g$  is locally integrable on  $I$ , then the dual action is well-defined. More generally, we will write  $\langle \cdot, \cdot \rangle_I$  to denote the dual action of distributions on test functions on the given interval  $I$ . Naturally, this agrees with the notation we have introduced so far for the case  $I = \mathbb{R}$ .

**7.6. The restriction of  $\mathfrak{L}(\mathbb{R})$  to an interval.** If  $u$  is a distribution on an open interval  $J$ , then the *restriction of  $u$  to an open subinterval  $I$* , denoted  $u|_I$ , is the distribution defined by

$$\langle \varphi, u|_I \rangle_I := \langle \varphi, u \rangle_J,$$

where  $\varphi$  is a  $C^\infty$ -smooth test function whose support is compact and contained in  $I$ .

**Definition 7.6.1.** Let  $I$  be an open interval of the real line. Then  $u \in \mathfrak{L}(I)$  means by definition that  $u$  is a distribution on  $I$  such that there exists a distribution  $v \in \mathfrak{L}(\mathbb{R})$  such that  $u = v|_I$ .

*Remark 7.6.2.* The following observation is pretty trivial, but quite useful. If we are given two open intervals  $I$  and  $J$  of the line  $\mathbb{R}$ , with  $I \subset J$ , then the restriction operation  $v \mapsto v|_I$  makes sense  $\mathfrak{L}(J) \rightarrow \mathfrak{L}(I)$ .

Proposition 7.4.5 has a localized version on a given interval  $J$ .

**Corollary 7.6.3.** *Suppose  $I, J \subset \mathbb{R}$  are open intervals with  $I \subset J$ . If  $u \in \mathfrak{L}(J)$  and  $\text{vp}[u] = 0$  almost everywhere on  $I$ , then the support of the distribution  $u$  has empty intersection with  $I$ .*

*Proof.* The assertion of the corollary is immediate from Proposition 7.4.5.  $\square$

The following result will prove quite useful.

**Proposition 7.6.4.** *Let  $I$  be a nonempty bounded open interval of the line  $\mathbb{R}$ . Then  $L^1(I)$  is a norm dense subspace of  $\mathfrak{L}(I)$ .*

*Proof.* As a matter of definition, we have that

$$\mathfrak{L}(I) = \mathfrak{L}(\mathbb{R})/\mathfrak{L}(\mathbb{R}; I),$$

where

$$\mathfrak{Z}(\mathbb{R}; I) := \{u \in \mathfrak{L}(\mathbb{R}) : I \cap \text{supp } u = \emptyset\}.$$

By elementary Functional Analysis, we know that the dual space  $\mathfrak{L}(I)^*$  is given by the annihilator

$$\mathfrak{L}(I)^* = \mathfrak{Z}(\mathbb{R}; I)^\perp = \{f \in H_\otimes^\infty(\mathbb{R}) : \forall u \in \mathfrak{Z}(\mathbb{R}; I) : \langle f, u \rangle_{\mathbb{R}} = 0\}.$$

**OBSERVATION.** We have that  $\mathfrak{Z}(\mathbb{R}; I)^\perp \subset \{f \in H_\otimes^\infty(\mathbb{R}) : f = 0 \text{ a.e. on } \mathbb{R} \setminus I\}$ .

**PROOF OF THE OBSERVATION:** Indeed, if  $f \in H_\otimes^\infty(\mathbb{R})$  and the restriction to  $\mathbb{R} \setminus I$  is nonzero on a set of positive Lebesgue measure, we readily construct a function  $u \in L^1(\mathbb{R})$  which vanishes on  $I$  such that  $\langle f, u \rangle_{\mathbb{R}} \neq 0$ . Since  $u \in \mathfrak{Z}(\mathbb{R}; I)$ , this proves the asserted containment.

We proceed with the proof of the proposition. If  $f \in H_\otimes^\infty(\mathbb{R})$  vanishes a.e. on  $\mathbb{R} \setminus I$ , and as a functional on  $\mathfrak{L}(I)$ ,  $f$  annihilates  $L^1(I)$ , then we may conclude that  $f = 0$  a.e. on  $I$  as well. But now  $f = 0$  a.e. on the line  $\mathbb{R}$ , so  $f = 0$  as an element of  $H_\otimes^\infty(\mathbb{R})$ . By the Hahn-Banach theorem, we derive that  $L^1(I)$  is norm dense in  $\mathfrak{L}(I)$ .  $\square$

*Remark 7.6.5.* A more refined argument shows that in the context of the observation, we actually have equality:  $\mathfrak{Z}(\mathbb{R}; I)^\perp = \{f \in H_\otimes^\infty(\mathbb{R}) : f = 0 \text{ a.e. on } \mathbb{R} \setminus I\}$ .

We may also translate Proposition 7.4.3 to this local context.

**Corollary 7.6.6.** *Let  $I$  be a nonempty open interval of the line  $\mathbb{R}$ . Then the “valeur au point” mapping is continuous  $\text{vp} : \mathfrak{L}(I) \rightarrow L^{1,\infty}(I)$ .*

## 8. BACKGROUND MATERIAL: THE HARDY AND BMO SPACES ON THE CIRCLE

**8.1. The Hardy  $H^1$  space on the circle: analytic and real.** Let  $L^1(\mathbb{R}/2\mathbb{Z})$  denote the space of 2-periodic Borel measurable functions  $f : \mathbb{R} \rightarrow \mathbb{C}$  subject to the integrability condition

$$\|f\|_{L^1(\mathbb{R}/2\mathbb{Z})} := \int_{I_1} |f(t)| dt < +\infty,$$

where  $I_1 = ]-1, 1[$  as before. As usual, we identify functions that agree except on a null set. Via the exponential mapping  $t \mapsto e^{i\pi t}$ , which is 2-periodic and maps the real line  $\mathbb{R}$  onto the unit circle  $\mathbb{T}$ , we may identify the space  $L^1(\mathbb{R}/2\mathbb{Z})$  with the standard Lebesgue space  $L^1(\mathbb{T})$  of the unit circle. This will allow us to develop the elements of Hardy space theory in the setting of 2-periodic functions. We shall need the subspace  $L_0^1(\mathbb{R}/2\mathbb{Z})$  consisting of all  $f \in L^1(\mathbb{R}/2\mathbb{Z})$  with

$$\langle f, 1 \rangle_{I_1} = \int_{I_1} f(t) dt = 0;$$

it has codimension 1 in  $L^1(\mathbb{R}/2\mathbb{Z})$ . The Hardy space  $H_+^1(\mathbb{R}/2\mathbb{Z})$  is defined as the subspace of  $L^1(\mathbb{R}/2\mathbb{Z})$  consisting of functions  $g \in L^1(\mathbb{R}/2\mathbb{Z})$  with

$$(8.1.1) \quad \int_{-1}^1 e^{i\pi n t} g(t) dt = 0, \quad n = 0, 1, 2, \dots$$

The space  $H_+^1(\mathbb{R}/2\mathbb{Z})$  is the periodic analogue of the Hardy space  $H_+^1(\mathbb{R})$ , and it can be understood in terms of the Hardy  $H^1$ -space of the disk. If  $H_+^1(\mathbb{T})$  denotes the standard Hardy space on the unit disk (restricted to the boundary unit circle), then  $g \in H_+^1(\mathbb{R}/2\mathbb{Z})$  means that  $g(x) = f(e^{i\pi x})$  for some  $f \in H_+^1(\mathbb{T})$  with  $f(0) = 0$ . In particular, the functions in  $H_+^1(\mathbb{R}/2\mathbb{Z})$  have holomorphic extensions to the upper half-plane which are 2-periodic. As a matter of definition,  $H_-^1(\mathbb{R}/2\mathbb{Z})$  consists of the functions  $g$  in  $L^1(\mathbb{R}/2\mathbb{Z})$  whose complex conjugate  $\bar{g}$  is in  $H_+^1(\mathbb{R}/2\mathbb{Z})$ . Finally, we put

$$H_\otimes^1(\mathbb{R}/2\mathbb{Z}) := H_+^1(\mathbb{R}/2\mathbb{Z}) \oplus H_-^1(\mathbb{R}/2\mathbb{Z}),$$

where we think of the elements of the sum space as 2-periodic functions (as before the symbol  $\oplus$  means direct sum, since  $H_+^1(\mathbb{R}/2\mathbb{Z}) \cap H_-^1(\mathbb{R}/2\mathbb{Z}) = \{0\}$ ). We note that, for instance,  $H_\otimes^1(\mathbb{R}/2\mathbb{Z}) \subset L_0^1(\mathbb{R}/2\mathbb{Z})$ . We will think of  $H_\otimes^1(\mathbb{R}/2\mathbb{Z})$  as the *real  $H^1$  space of 2-periodic functions*.



**8.2. The Hilbert transform on 2-periodic functions and distributions.** For  $f \in L^1(\mathbb{R}/2\mathbb{Z})$ , we let  $\mathbf{H}_2$  be the convolution operator

$$(8.2.1) \quad \mathbf{H}_2 f(x) := \frac{1}{2} \text{pv} \int_{I_1} f(t) \cot \frac{\pi(x-t)}{2} dt,$$

where again pv stands for principal value, which means we take the limit as  $\epsilon \rightarrow 0^+$  of the integral where the set

$$\{x\} + 2\mathbb{Z} + [-\epsilon, \epsilon]$$

is removed from the interval  $I_1 = ]-1, 1[$ . It is obvious from the periodicity of the cotangent function that  $\mathbf{H}_2 f$ , if it exists as a limit, is 2-periodic. By a standard trigonometric identity,

$$\frac{1}{2} \cot \frac{\pi y}{2} = \lim_{N \rightarrow +\infty} \frac{1}{\pi} \sum_{n=-N}^N \frac{1}{y + 2n},$$

where the convergence is uniform on compact subsets of the line. By a change of variables,

$$(8.2.2) \quad \mathbf{H}_2 f(x) = \frac{1}{2} \lim_{\epsilon \rightarrow 0^+} \int_{I_1 \setminus I_\epsilon} f(x-t) \cot \frac{\pi t}{2} dt,$$

(here, as usual,  $I_\epsilon = ]-\epsilon, \epsilon[$ ) from which we conclude, by uniform convergence and periodicity, that

$$\begin{aligned} (8.2.3) \quad \mathbf{H}_2 f(x) &= \frac{1}{\pi} \lim_{N \rightarrow +\infty} \lim_{\epsilon \rightarrow 0^+} \sum_{n=-N}^N \int_{I_1 \setminus I_\epsilon} f(x-t) \frac{dt}{t + 2n} \\ &= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0^+} \int_{I_1 \setminus I_\epsilon} f(x-t) \frac{dt}{t} + \frac{1}{\pi} \lim_{N \rightarrow +\infty} \sum_{n: |n| \leq N, n \neq 0} \int_{I_1} f(x-t) \frac{dt}{t + 2n} \\ &= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0^+} \int_{I_1 \setminus I_\epsilon} f(x-t) \frac{dt}{t} + \frac{1}{\pi} \lim_{N \rightarrow +\infty} \sum_{n: |n| \leq N, n \neq 0} \int_{[2n-1, 2n+1]} f(x-t) \frac{dt}{t} \\ &= \lim_{N \rightarrow +\infty} \lim_{\epsilon \rightarrow 0^+} \frac{1}{\pi} \int_{I_{2N+1} \setminus I_\epsilon} f(x-t) \frac{dt}{t}. \end{aligned}$$

In other words, the operator  $\mathbf{H}_2$  is just the natural extension of the Hilbert transform to the 2-periodic functions. We observe that  $\mathbf{H}_2 1 = 0$ , which contrasts with the non-periodic case (where no nontrivial function is mapped to the zero function). It is well-known that the periodic Hilbert transform  $\mathbf{H}_2$  maps  $L^1(\mathbb{R}/2\mathbb{Z})$  into the weak  $L^1$ -space  $L^{1,\infty}(\mathbb{R}/2\mathbb{Z})$ . However, we prefer to work within the framework of distribution theory, so we proceed as follows.

Let  $C^\infty(\mathbb{R}/2\mathbb{Z})$  denote the space of  $C^\infty$ -smooth 2-periodic functions on  $\mathbb{R}$ . It is easy to see that

$$\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z}) \implies \mathbf{H}_2 \varphi \in C^\infty(\mathbb{R}/2\mathbb{Z}).$$

To emphasize the importance of the circle  $\mathbb{T} \cong \mathbb{R}/2\mathbb{Z}$ , we write

$$(8.2.4) \quad \langle f, g \rangle_{\mathbb{R}/2\mathbb{Z}} := \int_{-1}^1 f(t) g(t) dt,$$

for the dual action when  $f$  and  $g$  are 2-periodic.

**Definition 8.2.1.** For a test function  $\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z})$  and a distribution  $u$  on the circle  $\mathbb{R}/2\mathbb{Z}$ , we put

$$\langle \varphi, \mathbf{H}_2 u \rangle_{\mathbb{R}/2\mathbb{Z}} := -\langle \mathbf{H}_2 \varphi, u \rangle_{\mathbb{R}/2\mathbb{Z}}.$$

This defines the Hilbert transform  $\mathbf{H}_2 u$  for any distribution  $u$  on the circle  $\mathbb{R}/2\mathbb{Z}$ .

The analogue of Proposition 7.1.1 for the circle reads as follows. Note that the formula definition of the “valeur au point” function makes sense also for  $u$  in the space of distributions  $L^1(\mathbb{R}/2\mathbb{Z}) + \mathbf{H}_2 L^1(\mathbb{R}/2\mathbb{Z})$ . Moreover, the independence of the cut-off function is quite analogous to the real line case (Lemma 7.4.2) and left to the interested reader.

**Proposition 8.2.2.** *Suppose  $f \in L^1_0(\mathbb{R}/2\mathbb{Z})$ . Then the following are equivalent:*

- (i)  $f \in H^1_\otimes(\mathbb{R}/2\mathbb{Z})$ .
- (ii)  $\mathbf{H}_2 f \in L^1(\mathbb{R}/2\mathbb{Z})$ , where  $\mathbf{H}_2 f$  is understood as a distribution on the line  $\mathbb{R}$ .
- (iii)  $\text{vp}[\mathbf{H}_2 f] \in L^1(\mathbb{R}/2\mathbb{Z})$ .

*Proof.* This is immediate from [20], p. 87.  $\square$

**8.3. The real  $H^\infty$ -space of the circle.** The real  $H^\infty$ -space on the circle  $\mathbb{R}/2\mathbb{Z}$  is denoted by  $H^\infty_\otimes(\mathbb{R}/2\mathbb{Z})$ , and consists of all the functions in  $H^\infty_\otimes(\mathbb{R})$  that are 2-periodic. The analogue of Proposition 7.2.1 reads:

**Proposition 8.3.1.** *We have the equivalence*

$$f \in H^\infty_\otimes(\mathbb{R}/2\mathbb{Z}) \iff f, \mathbf{H}_2 f \in L^\infty(\mathbb{R}/2\mathbb{Z}).$$

This result is well-known.

**8.4. A predual of 2-periodic real  $H^\infty$ .** We put

$$\mathfrak{L}(\mathbb{R}/2\mathbb{Z}) := L^1(\mathbb{R}/2\mathbb{Z}) + \mathbf{H}_2 L^1_0(\mathbb{R}/2\mathbb{Z}),$$

understood as a space of 2-periodic distributions on the line  $\mathbb{R}$ . More precisely, if  $u = f + \mathbf{H}_2 g$ , where  $f \in L^1(\mathbb{R}/2\mathbb{Z})$  and  $g \in L^1_0(\mathbb{R}/2\mathbb{Z})$ , then the action on a test function  $\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z})$  is given by

$$(8.4.1) \quad \langle \varphi, u \rangle_{\mathbb{R}/2\mathbb{Z}} := \langle \varphi, f \rangle_{\mathbb{R}/2\mathbb{Z}} - \langle \mathbf{H}_2 \varphi, g \rangle_{\mathbb{R}/2\mathbb{Z}}.$$

But a 2-periodic distribution should be possible to think of as a distribution on the line, which means that need to understand the action on standard test functions in  $C^\infty_c(\mathbb{R})$ . If  $\psi \in C^\infty_c(\mathbb{R})$ , we simply put

$$(8.4.2) \quad \langle \psi, u \rangle_{\mathbb{R}/2\mathbb{Z}} := \langle \Pi_2 \psi, u \rangle_{\mathbb{R}/2\mathbb{Z}},$$

where  $\Pi_2 \psi \in C^\infty(\mathbb{R}/2\mathbb{Z})$  is given by

$$(8.4.3) \quad \Pi_2 \psi(x) := \sum_{j \in \mathbb{Z}} \psi(x + 2j).$$

We will refer to  $\Pi_2$  as the *periodization operator*.

As in the case of the line, we may identify  $\mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  with the predual of the real  $H^\infty$  space.

**Proposition 8.4.1.** *Each continuous linear functional  $\mathfrak{L}(\mathbb{R}/2\mathbb{Z}) \rightarrow \mathbb{C}$  corresponds to a function  $\varphi \in H^\infty_\otimes(\mathbb{R}/2\mathbb{Z})$  in accordance with (8.4.1). In short, the dual space of  $\mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  is isomorphic to  $H^\infty_\otimes(\mathbb{R}/2\mathbb{Z})$ .*

We suppress the proof, which is analogous to that of Proposition 7.3.1.

The definition of the “valeur au point”  $\text{vp}[u]$  makes sense for  $u \in \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  and as in the case of the line, it does not depend on the choice of the particular cut-off function. We have the analogue of Proposition 7.4.3; as the result is standard, we suppress the proof.

**Proposition 8.4.2.** (Kolmogorov) *The “valeur au point” mapping  $\text{vp} : \mathfrak{L}(\mathbb{R}/2\mathbb{Z}) \rightarrow L^{1,\infty}(\mathbb{R}/2\mathbb{Z})$ ,  $u \mapsto \text{vp}[u]$ , is continuous.*

## 9. A SUM OF TWO PREDUALS AND ITS LOCALIZATION TO INTERVALS

**9.1. The sum space  $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$ .** Suppose  $u$  is distribution on the line  $\mathbb{R}$  of the form

$$(9.1.1) \quad u = v + w, \quad \text{where } v \in \mathfrak{L}(\mathbb{R}), \quad w \in \mathfrak{L}(\mathbb{R}/2\mathbb{Z}).$$

The natural question appears as to whether the distributions  $v, w$  on the right hand side are unique. This is indeed so.

**Proposition 9.1.1.** *We have that  $\mathfrak{L}(\mathbb{R}) \cap \mathfrak{L}(\mathbb{R}/2\mathbb{Z}) = \{0\}$ .*

This last statement is pretty obvious in terms of the Fourier transform, which sends 2-periodic distributions to sums of point masses along the integers, while the space  $\mathfrak{L}(\mathbb{R})$  is mapped to a space of bounded continuous functions.

In view of Proposition 9.1.1, it makes sense to write  $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  for the space of tempered distributions  $u$  of the form (9.1.1). We endow  $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  with the induced Banach space norm

$$\|u\|_{\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})} := \|v\|_{\mathfrak{L}(\mathbb{R})} + \|w\|_{\mathfrak{L}(\mathbb{R}/2\mathbb{Z})},$$

provided  $u, v, w$  are related via (9.1.1).

**9.2. The localization of  $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  to a bounded open interval.** In the sense of Subsection 7.6, we may restrict a given distribution  $u \in \mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  to a given open interval  $I$ . It is natural to wonder what the space of such restrictions looks like.

**Proposition 9.2.1.** *The restriction of the space  $\mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  to a bounded open interval  $I$  equals the space  $\mathfrak{L}(I)$ .*

*Proof.* By definition, the restriction of  $\mathfrak{L}(\mathbb{R})$  to  $I$  equals  $\mathfrak{L}(I)$ . It remains to show that the restriction to  $I$  of a distribution in  $\mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  is in  $\mathfrak{L}(I)$  as well. Since

$$\mathfrak{L}(\mathbb{R}/2\mathbb{Z}) = L^1(\mathbb{R}/2\mathbb{Z}) + \mathbf{H}_2 L_0^1(\mathbb{R}/2\mathbb{Z}),$$

and the restriction to the bounded interval  $I$  of  $L^1(\mathbb{R}/2\mathbb{Z})$  is contained in  $L^1(I)$ , the only thing we need to check is that the restriction of  $\mathbf{H}_2 L_0^1(\mathbb{R}/2\mathbb{Z})$  to  $I$  is contained in  $\mathfrak{L}(I)$ . It will be enough to show that for each  $f \in L_0^1(\mathbb{R}/2\mathbb{Z})$ , there exist  $g \in L^1(\mathbb{R})$ ,  $h \in L_0^1(\mathbb{R})$ , and a distribution  $W \in \mathcal{D}'(\mathbb{R})$  with support contained in  $\mathbb{R} \setminus I$ , such that

$$\mathbf{H}_2 f = g + \mathbf{H}h + W.$$

We need two bounded open intervals  $J_1, J_2$  such that  $I \Subset J_1 \Subset J_2$ . We first let  $h$  equal  $f$  on  $J_1$ , and put it equal to 0 on  $\mathbb{R} \setminus J_2$ . In the difference set  $J_2 \setminus J_1$ , we let  $h$  be constant, where the value of the constant is then determined by the requirement that  $h \in L_0^1(\mathbb{R})$ . As the cotangent kernel  $\frac{1}{2} \cot \frac{\pi t}{2}$  used to define  $\mathbf{H}_2$  and the Hilbert transform kernel  $\frac{1}{\pi t}$  have the same singularity, it is easy to see that  $\mathbf{H}_2 f - \mathbf{H}h$  is smooth on  $J_1$ , and we may declare  $g$  to equal  $\mathbf{H}_2 f - \mathbf{H}h$  on  $I$ , and put it equal to 0 on the rest  $\mathbb{R} \setminus I$ . The distribution  $W$  is uniquely determined by these choices, and has the required properties.  $\square$

## 10. AN INVOLUTION, ITS ADJOINT, AND THE PERIODIZATION OPERATOR

**10.1. An involutive operator.** For each positive real number  $\beta$ , let  $\mathbf{J}_\beta$  denote the involution given by

$$\mathbf{J}_\beta f(x) := \frac{\beta}{x^2} f(-\beta/x), \quad x \in \mathbb{R}^\times.$$

With respect to the dual action  $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ , this operator  $\mathbf{J}_\beta$  can be understood as the preadjoint of the involution  $\mathbf{J}_\beta^*$  defined in (5.1.1).

As usual, we use the standard notation  $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$ . We now record some basic properties of this involution. For instance, by the change-of-variables formula,  $\mathbf{J}_\beta : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$  is an *isometry*.

**Proposition 10.1.1.** *Fix  $0 < \beta < +\infty$ . The operator  $\mathbf{J}_\beta$  is an isometric isomorphism  $L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ . In addition,  $\mathbf{J}_\beta$  maps  $H_+^1(\mathbb{R}) \rightarrow H_+^1(\mathbb{R})$  and  $H_-^1(\mathbb{R}) \rightarrow H_-^1(\mathbb{R})$  and consequently  $\mathbf{J}_\beta : H_\otimes^1(\mathbb{R}) \rightarrow H_\otimes^1(\mathbb{R})$  as well.*

*Proof.* The mapping  $z \mapsto -\beta/z$  preserves the upper half-plane  $\mathbb{C}_+$ , and so that functions holomorphic in  $\mathbb{C}_+$  are sent to functions holomorphic in  $\mathbb{C}_+$  under composition by  $z \mapsto -\beta/z$ . The isometric part is already settled, so it remains to check that the space  $H_+^1(\mathbb{R})$  is preserved under  $\mathbf{J}_\beta$ , since the case of  $H_-^1(\mathbb{R})$  is identical. This follows easily by checking the property on a dense subspace (e.g. consisting of rational functions).  $\square$

If  $f \in L^1(\mathbb{R})$  and  $\varphi \in L^\infty(\mathbb{R})$ , the change-of-variables formula yields

$$(10.1.1) \quad \langle \varphi, \mathbf{J}_\beta f \rangle_{\mathbb{R}} = \int_{\mathbb{R}} \varphi(t) f(-\beta/t) \frac{\beta dt}{t^2} = \int_{\mathbb{R}} \varphi(-\beta/t) f(t) dt = \langle \mathbf{J}_\beta^* \varphi, f \rangle_{\mathbb{R}},$$

where  $\mathbf{J}_\beta^*$  is the involution

$$\mathbf{J}_\beta^* \varphi(t) := \varphi(-\beta/t), \quad t \in \mathbb{R}^\times.$$

We need to extend  $\mathbf{J}_\beta$  to an operator  $\mathfrak{V}(\mathbb{R}) \rightarrow \mathfrak{V}(\mathbb{R})$ . To this end, we need to understand how to define  $\mathbf{J}_\beta \mathbf{H}f$  as a distribution in  $\mathfrak{V}(\mathbb{R})$  when  $f \in L_0^1(\mathbb{R})$ . First, following (10.1.1), we put

$$(10.1.2) \quad \langle \varphi, \mathbf{J}_\beta \mathbf{H}f \rangle_{\mathbb{R}} := -\langle \mathbf{H} \mathbf{J}_\beta^* \varphi, f \rangle_{\mathbb{R}},$$

for  $f \in L_0^1(\mathbb{R})$  and for test functions  $\varphi \in C_c^\infty(\mathbb{R}^\times)$ , since such test functions vanish near the origin. Note here that if  $\varphi \in C_c^\infty(\mathbb{R}^\times)$ , then necessarily  $\mathbf{J}_\beta^* \varphi \in C_c^\infty(\mathbb{R}^\times)$  as well, so the right-hand side of (10.1.2) is well-defined.

**Proposition 10.1.2.** *For a test function  $\varphi \in C_c^\infty(\mathbb{R}^\times)$ , we have the identity*

$$\mathbf{H} \mathbf{J}_\beta^* \varphi(x) = \mathbf{J}_\beta^* \mathbf{H} \varphi(x) - \langle \varphi, t \mapsto \frac{1}{\pi t} \rangle_{\mathbb{R}}, \quad x \in \mathbb{R}^\times.$$

*Proof.* By a change of variables in the corresponding integral, we have that

$$\mathbf{J}_\beta^* \mathbf{H} \varphi(x) = -\frac{1}{\pi} \text{pv} \int_{\mathbb{R}} \frac{x}{\beta + tx} \varphi(t) dt, \quad \mathbf{H} \mathbf{J}_\beta^* \varphi(x) = \frac{1}{\pi} \text{pv} \int_{\mathbb{R}} \frac{\beta}{t(\beta + tx)} \varphi(t) dt,$$

so the asserted equality is a simple consequence of the algebraic identity

$$-\frac{x}{\beta + tx} = \frac{\beta}{t(\beta + tx)} - \frac{1}{t}.$$

The proof is complete.  $\square$

As  $f \in L_0^1(\mathbb{R})$ , its action on constants vanishes, so by a combination of (10.1.1), (10.1.2), and Proposition 10.1.2, we obtain

$$(10.1.3) \quad \langle \varphi, \mathbf{J}_\beta \mathbf{H}f \rangle_{\mathbb{R}} = -\langle \mathbf{H} \mathbf{J}_\beta^* \varphi, f \rangle_{\mathbb{R}} = -\langle \mathbf{J}_\beta^* \mathbf{H} \varphi, f \rangle_{\mathbb{R}} = -\langle \mathbf{H} \varphi, \mathbf{J}_\beta f \rangle_{\mathbb{R}} = \langle \varphi, \mathbf{H} \mathbf{J}_\beta f \rangle_{\mathbb{R}},$$

for  $\varphi \in C_c^\infty(\mathbb{R}^\times)$ . As  $f \in L_0^1(\mathbb{R})$ , we also have that  $\mathbf{J}_\beta f \in L_0^1(\mathbb{R})$ , so that  $\mathbf{H} \mathbf{J}_\beta f \in \mathbf{H} L_0^1(\mathbb{R}) \subset \mathfrak{V}(\mathbb{R})$ . This means that as distributions on the punctured line  $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ ,  $\mathbf{J}_\beta \mathbf{H}f$  and  $\mathbf{H} \mathbf{J}_\beta f$  coincide. In particular, their “valeur au point” functions, which are well-defined almost everywhere, coincide on  $\mathbb{R}^\times$ . However, the distribution  $\mathbf{H} \mathbf{J}_\beta f$  makes sense on test functions  $\varphi \in C_c^\infty(\mathbb{R})$ , and actually, more generally for  $\varphi \in H_\otimes^\infty(\mathbb{R})$ . This allows us to extend the action of  $\mathbf{J}_\beta \mathbf{H}f$  from  $C_c^\infty(\mathbb{R}^\times)$  to  $H_\otimes^\infty(\mathbb{R})$  (compare with (7.3.2)).

**Definition 10.1.3.** For  $u \in \mathfrak{V}(\mathbb{R})$  of the form  $u = f + \mathbf{H}g \in \mathfrak{V}(\mathbb{R})$ , where  $f \in L^1(\mathbb{R})$  and  $g \in L_0^1(\mathbb{R})$ , we define the  $\mathbf{J}_\beta u$  to be the distribution on  $\mathbb{R}$  given by the formula

$$\langle \varphi, \mathbf{J}_\beta u \rangle_{\mathbb{R}} = \langle \varphi, \mathbf{J}_\beta (f + \mathbf{H}g) \rangle_{\mathbb{R}} := \langle \varphi, \mathbf{J}_\beta f \rangle_{\mathbb{R}} + \langle \varphi, \mathbf{H} \mathbf{J}_\beta g \rangle_{\mathbb{R}} = \langle \varphi, \mathbf{J}_\beta f \rangle_{\mathbb{R}} - \langle \tilde{\mathbf{H}} \varphi, \mathbf{J}_\beta g \rangle_{\mathbb{R}},$$

for test functions  $\varphi \in H_\otimes^\infty(\mathbb{R})$ .

As already noted, this is in complete agreement with the way we would previously understand  $\mathbf{J}_\beta u$  as a distribution on  $\mathbb{R}^\times$ , using smooth test functions having compact support on the punctured line  $\mathbb{R}^\times$ ; see (10.1.1) and (10.1.2).

**Proposition 10.1.4.** *Fix  $0 < \beta < +\infty$ . The involution  $\mathbf{J}_\beta$  acts continuously  $\mathfrak{V}(\mathbb{R}) \rightarrow \mathfrak{V}(\mathbb{R})$ , and the involution  $\mathbf{J}_\beta^*$  acts continuously  $H_\otimes^\infty(\mathbb{R}) \rightarrow H_\otimes^\infty(\mathbb{R})$ . Moreover, on their respective spaces,  $\mathbf{J}_\beta^2$  and  $\mathbf{J}_\beta^{*2}$  both equal the identity operator.*

*Proof.* Let  $u \in \mathfrak{L}(\mathbb{R})$  be of the form  $u = f + \mathbf{H}g$ , where  $f \in L^1(\mathbb{R})$  and  $g \in L_0^1(\mathbb{R})$ . Then, by definition,  $\mathbf{J}_\beta u = \mathbf{J}_\beta f + \mathbf{H}\mathbf{J}_\beta g \in \mathfrak{L}(\mathbb{R})$ , and it is clear that the mapping  $\mathbf{J}_\beta$  acts continuously. Moreover, by iteration

$$\mathbf{J}_\beta^2 u = \mathbf{J}_\beta^2 f + \mathbf{H}\mathbf{J}_\beta^2 g = f + \mathbf{H}g = u$$

since  $\mathbf{J}_\beta^2 F = F$  holds for all  $F \in L^1(\mathbb{R})$ . The assertions concerning the adjoint  $\mathbf{J}_\beta^*$  follow by duality.  $\square$

**10.2. The periodization operator.** We recall the definition of the *periodization operator*  $\Pi_2$ :

$$\Pi_2 f(x) := \sum_{j \in \mathbb{Z}} f(x + 2j).$$

In (8.4.3), we defined the  $\Pi_2$  on test functions. It is however clear that it remains well-defined with much less smoothness required of  $f$ . The terminology comes from the property that whenever it is well-defined, the function  $\Pi_2 f$  is 2-periodic automatically. A first result is the following.

**Proposition 10.2.1.** *The operator  $\Pi_2$  acts contractively  $L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R}/2\mathbb{Z})$ . Moreover,  $\Pi_2$  maps  $H_+^1(\mathbb{R})$  onto  $H_+^1(\mathbb{R}/2\mathbb{Z})$  and  $H_-^1(\mathbb{R})$  onto  $H_-^1(\mathbb{R}/2\mathbb{Z})$ .*

*Proof.* By the triangle inequality and Fubini's theorem,  $\Pi_2$  is a contraction  $L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R}/2\mathbb{Z})$ :

$$\int_{-1}^1 |\Pi_2 f(x)| dx \leq \sum_{j \in \mathbb{Z}} \int_{-1}^1 |f(x + 2j)| dx = \sum_{j \in \mathbb{Z}} \int_{2j-1}^{2j+1} |f(x)| dx = \int_{\mathbb{R}} |f(x)| dx,$$

It remains to check the mapping properties, which are immediate from the characterizations (4.1.3), (4.1.4) for the line and (8.1.1) for the circle, combined with the calculation

$$(10.2.1) \quad \int_{-1}^1 e^{i\pi n t} \Pi_2 f(t) dt = \sum_{j \in \mathbb{Z}} \int_{-1}^1 e^{i\pi n t} f(t + 2j) dt = \int_{\mathbb{R}} e^{i\pi n t} f(t) dt, \quad n \in \mathbb{Z}.$$

The proof is complete.  $\square$

The identity (10.2.1) is a special case of a more general identity, for  $f \in L^1(\mathbb{R})$  and  $F \in L^\infty(\mathbb{R}/2\mathbb{Z})$  (compare with (8.4.2)):

$$(10.2.2) \quad \langle F, \Pi_2 f \rangle_{\mathbb{R}/2\mathbb{Z}} = \int_{-1}^1 F(t) \Pi_2 f(t) dt = \sum_{j \in \mathbb{Z}} \int_{-1}^1 F(t) f(t + 2j) dt \\ = \int_{\mathbb{R}} F(t) f(t) dt = \langle F, f \rangle_{\mathbb{R}}, \quad n \in \mathbb{Z}.$$

We need to extend  $\Pi_2$  in a natural fashion to the space  $\mathfrak{L}(\mathbb{R})$ . If  $\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z})$  is a test function on the circle, we glance at (10.2.2), and for  $u \in \mathfrak{L}(\mathbb{R})$  with  $u = f + \mathbf{H}g$ , where  $f \in L^1(\mathbb{R})$  and  $g \in L_0^1(\mathbb{R})$ , we set

$$(10.2.3) \quad \langle \varphi, \Pi_2 u \rangle_{\mathbb{R}/2\mathbb{Z}} := \langle \varphi, u \rangle_{\mathbb{R}} = \langle \varphi, f \rangle_{\mathbb{R}} - \langle \tilde{\mathbf{H}}\varphi, g \rangle_{\mathbb{R}}.$$

This defines  $\Pi_2 u$  as a distribution on the circle (compare with (7.3.2)).

**Proposition 10.2.2.** *For  $u \in \mathfrak{L}(\mathbb{R})$  of the form  $u = f + \mathbf{H}g$ , where  $f \in L^1(\mathbb{R})$  and  $g \in L_0^1(\mathbb{R})$ , we have that  $\Pi_2 u = \Pi_2 f + \mathbf{H}_2 \Pi_2 g$ . In particular,  $\Pi_2$  maps  $\mathfrak{L}(\mathbb{R}) \rightarrow \mathfrak{L}(\mathbb{R}/2\mathbb{Z})$  continuously.*

*Proof.* For a 2-periodic test function  $\varphi \in C^\infty(\mathbb{R}/2\mathbb{Z})$ , we check that

$$\langle \varphi, \Pi_2 f + \mathbf{H}_2 \Pi_2 g \rangle_{\mathbb{R}/2\mathbb{Z}} = \langle \varphi, \Pi_2 f \rangle_{\mathbb{R}/2\mathbb{Z}} - \langle \mathbf{H}_2 \varphi, \Pi_2 g \rangle_{\mathbb{R}/2\mathbb{Z}} = \langle \varphi, f \rangle_{\mathbb{R}} - \langle \mathbf{H}_2 \varphi, g \rangle_{\mathbb{R}},$$

where we applied the identity (10.2.2) twice. If we compare this with (10.2.3), we realize we have the same expression, because  $\tilde{\mathbf{H}}\varphi$  and  $\mathbf{H}_2 \varphi$  differ by a constant. After all, they are two harmonic conjugates of one and the same function, and  $g$  annihilates constants.  $\square$

## 11. THE SPANNING PROBLEM FORMULATION OF THEOREM 1.8.2

11.1. **A reformulation of Theorem 1.8.2.** Let us consider the following problem.

**Problem 11.1.1.** For which values of the positive real parameter  $\beta$  is the linear span of the functions

$$e_n(t) := e^{i\pi n t}, \quad e_m^{(\beta)}(t) := e^{-i\pi\beta m/t}, \quad m, n \in \mathbb{Z}_{+,0},$$

weak-star dense in  $H_+^\infty(\mathbb{R})$ ?

We first remark that the functions  $e^{i\pi n t}$  and  $e^{-i\pi\beta m/t}$  for  $m, n \in \mathbb{Z}_{+,0}$  belong to  $H_+^\infty(\mathbb{R})$  (they have bounded holomorphic extensions to  $\mathbb{C}_+$ ), so that the problem makes sense. A simple scaling argument allows us to take  $\alpha := 1$ , so that *Theorem 1.8.2 is equivalent to Problem 11.1.1 having an affirmative answer if and only if  $\beta \leq 1$ .*

With respect to the dual action  $\langle \cdot, \cdot \rangle_{\mathbb{R}}$  on the line, the understood predual of  $H_+^\infty(\mathbb{R})$  is the quotient space  $L^1(\mathbb{R})/H_+^1(\mathbb{R})$ . So, in terms of duality, the question raised in Problem 11.1.1 is: *When, provided that  $f \in L^1(\mathbb{R})$ , do we have the implication*

$$(11.1.1) \quad \langle e_n, f \rangle_{\mathbb{R}} = \langle e_m^{(\beta)}, f \rangle_{\mathbb{R}} = 0 \quad \forall m, n \in \mathbb{Z}_{+,0} \implies f \in H_+^1(\mathbb{R})?$$

The argument involving point separation in  $\mathbb{C}_+$  from [16] applies here as well, which makes  $\beta \leq 1$  a necessary condition for the implication (11.1.1) to hold. Actually, as mentioned in the introduction, the methods of [7] supply infinitely many linearly independent counterexamples for  $\beta > 1$ .

Also, by testing with  $n = 0$ , we note that we might as well assume that  $f \in L_0^1(\mathbb{R})$  in (11.1.1). In view of (10.2.1),

$$(11.1.2) \quad \langle e_n, f \rangle_{\mathbb{R}} = \int_{-1}^1 e^{i\pi n t} \Pi_2 f(t) dt = \langle e_n, \Pi_2 f \rangle_{\mathbb{R}/2\mathbb{Z}},$$

so that for  $f \in L^1(\mathbb{R})$  we have the equivalence

$$\left\{ \forall n \in \mathbb{Z}_{+,0} : \langle e_n, f \rangle_{\mathbb{R}} = 0 \right\} \iff \Pi_2 f \in H_+^1(\mathbb{R}/2\mathbb{Z}).$$

Since  $\mathbf{J}_\beta^* e_m = e_m^{(\beta)}$ , where  $\mathbf{J}_\beta^*$  is the involutive operator studied in Subsections 5.1 and 10.1, we have that

$$\langle f, e_m^{(\beta)} \rangle_{\mathbb{R}} = \langle f, \mathbf{J}_\beta^* e_m \rangle_{\mathbb{R}} = \langle \mathbf{J}_\beta f, e_m \rangle_{\mathbb{R}},$$

which leads for  $f \in L^1(\mathbb{R})$  to the equivalence

$$\left\{ \forall m \in \mathbb{Z}_{+,0} : \langle e_m^{(\beta)}, f \rangle_{\mathbb{R}} = 0 \right\} \iff \Pi_2 \mathbf{J}_\beta f \in H_+^1(\mathbb{R}/2\mathbb{Z}).$$

We can now rephrase the question (11.1.1) and hence Problem 11.1.1.

**Problem 11.1.2.** Fix  $0 < \beta \leq 1$ . Is it true that for  $f \in L_0^1(\mathbb{R})$ ,

$$\Pi_2 f, \Pi_2 \mathbf{J}_\beta f \in H_+^1(\mathbb{R}/2\mathbb{Z}) \implies f \in H_+^1(\mathbb{R})?$$

It is rather obvious that the reverse implication holds (use, e.g., Propositions 10.1.1 and 10.2.1). If we think of  $\Pi_2 f$  and  $\Pi_2 \mathbf{J}_\beta f$  as 2-periodic “shadows” of  $f$  and  $\mathbf{J}_\beta f$ , the issue at hand is whether knowing that the two shadows are in the right space we may conclude the function comes from the space  $H_+^1(\mathbb{R})$ . We note here that the main result of [16] may be understood as the assertion that  *$f$  is determined uniquely by the two “shadows”  $\Pi_2 f$  and  $\Pi_2 \mathbf{J}_\beta f$  if and only if  $\beta \leq 1$ .* This offers some rather weak support for the plausibility of the implication of Problem 11.1.2.

**11.2. An alternative reformulation in terms of the space  $\mathfrak{L}(\mathbb{R})$ .** We begin with a function  $f \in L_0^1(\mathbb{R})$ , and form the conjugate-analytic Szegő projection (cf. (4.2.6))

$$u := \mathbf{P}_- f = \frac{1}{2}(f - i\mathbf{H}f) \in L_0^1(\mathbb{R}) + \mathbf{H}L_0^1(\mathbb{R}) \subset \mathfrak{L}(\mathbb{R}).$$

Then, by Definition 10.1.3,

$$\mathbf{J}_\beta u = \mathbf{J}_\beta \mathbf{P}_- f = \frac{1}{2}(\mathbf{J}_\beta f - i\mathbf{J}_\beta \mathbf{H}f) = \frac{1}{2}(\mathbf{J}_\beta f - i\mathbf{H}\mathbf{J}_\beta f) \in L_0^1(\mathbb{R}) + \mathbf{H}L_0^1(\mathbb{R}) \subset \mathfrak{L}(\mathbb{R}),$$

and we calculate that (use Lemma 10.2.2)

$$(11.2.1) \quad \Pi_2 u = \frac{1}{2}(\Pi_2 f - i\Pi_2 \mathbf{H}f) = \frac{1}{2}(\Pi_2 f - i\mathbf{H}_2 \Pi_2 f) = \frac{1}{2}(\mathbf{I} - i\mathbf{H}_2)\Pi_2 f \in \mathfrak{L}(\mathbb{R}/2\mathbb{Z}),$$

and that (use Proposition 10.2.2 again)

$$(11.2.2) \quad \Pi_2 \mathbf{J}_\beta u = \frac{1}{2}(\Pi_2 \mathbf{J}_\beta f - i\Pi_2 \mathbf{H}\mathbf{J}_\beta f) = \frac{1}{2}(\Pi_2 \mathbf{J}_\beta f - i\mathbf{H}_2 \Pi_2 \mathbf{J}_\beta f) = \frac{1}{2}(\mathbf{I} - i\mathbf{H}_2)\Pi_2 \mathbf{J}_\beta f \in \mathfrak{L}(\mathbb{R}/2\mathbb{Z}).$$

Here, we write  $\mathbf{I}$  for the identity operator. Modulo the constants, the operator  $\mathbf{P}_{2,-} := \frac{1}{2}(\mathbf{I} - i\mathbf{H}_2)$  projects to the 2-periodic conjugate-holomorphic functions in the upper half-plane  $\mathbb{C}_+$ , and  $H_+^1(\mathbb{R}/2\mathbb{Z})$  is indeed mapped to  $\{0\}$ :

$$(11.2.3) \quad \mathbf{P}_{2,-} H_+^1(\mathbb{R}/2\mathbb{Z}) = \{0\}.$$

Hence we conclude from (11.2.1) and (11.2.2) that

$$\Pi_2 f, \Pi_2 \mathbf{J}_\beta f \in H_+^1(\mathbb{R}/2\mathbb{Z}) \implies \Pi_2 u = \Pi_2 \mathbf{J}_\beta u = 0.$$

We are led to consider the following problem. Let  $\mathfrak{L}_0(\mathbb{R})$  be the one-codimensional subspace of  $\mathfrak{L}(\mathbb{R})$  given by

$$\mathfrak{L}_0(\mathbb{R}) := L_0^1(\mathbb{R}) + \mathbf{H}L_0^1(\mathbb{R}) \subset \mathfrak{L}(\mathbb{R}).$$

**Problem 11.2.1.** Fix  $0 < \beta \leq 1$ . Is it true that for  $u \in \mathfrak{L}_0(\mathbb{R})$ ,

$$\Pi_2 u = \Pi_2 \mathbf{J}_\beta u = 0 \implies u = 0?$$

**Proposition 11.2.2.** *If the answer to Problem 11.2.1 is affirmative, then the answers to Problems 11.1.1 and 11.1.2 are affirmative as well, and the assertion of Theorem 1.8.2 is valid.*

*Proof.* We already know that Problems 11.1.1 and 11.1.2 are equivalent. Let  $f \in L^1(\mathbb{R})$  be such that  $\Pi_2 f \in H_+^1(\mathbb{R}/2\mathbb{Z})$  and  $\Pi_2 \mathbf{J}_\beta f \in H_+^1(\mathbb{R}/2\mathbb{Z})$ . Then, as a first step,  $f \in L_0^1(\mathbb{R})$  by the identity (10.2.1) with  $n = 0$ . We recall the notation  $\mathbf{P}_- := \frac{1}{2}(\mathbf{I} - i\mathbf{H})$  for the Szegő projection to the conjugate-holomorphic functions in  $\mathbb{C}_+$ . Next, we consider the distribution  $u := \mathbf{P}_- f = \frac{1}{2}(f - i\mathbf{H}f) \in \mathfrak{L}_0(\mathbb{R})$ , and use the identities (11.2.1) and (11.2.2) together with (11.2.3) to see that  $\Pi_2 u = \Pi_2 \mathbf{J}_\beta u = 0$ . Now, given that Problem 11.2.1 has an affirmative answer, we have that  $\mathbf{P}_- f = u = 0$ , which is only possible for  $f \in L^1(\mathbb{R})$  if  $f \in H_+^1(\mathbb{R})$ . We conclude that Problems 11.1.1 and 11.1.2 have affirmative answers as well. Finally, given the discussion in Subsection 1.8, the correctness of the assertion of Theorem 1.8.2 follows as well.  $\square$

**11.3. The connection with an extension of ergodic theory.** In [17], the following result is obtained as an application of an extension of ergodic theory in the setting of Gauss-type maps.

**Theorem 11.3.1.** (see [17]) *For  $0 < \beta \leq 1$  and  $u \in \mathfrak{L}_0(\mathbb{R})$ , the following implication holds:*

$$\Pi_2 u = \Pi_2 \mathbf{J}_\beta u = 0 \implies u = 0.$$

Modulo this result, we may now conclude the proof of Theorem 1.8.2.

*Proof of Theorem 1.8.2.* As observed right after the formulation of Theorem 1.8.2, a scaling argument allows us to reduce the redundancy and fix  $\alpha = 1$ , in which case the condition  $0 < \alpha \leq 1$  reads  $0 < \beta \leq 1$ . Now, in view of the above Subsection 11.1 and ensuing Proposition 11.2.2, the required assertion is an immediate consequence of Theorem 11.3.1.  $\square$

It remains to explain how Theorem 11.3.1 connects with an extension of ergodic theory. The connection is strongest for  $\beta = 1$ , which is why we restrict our attention to this value of  $\beta$ . For  $u \in \mathfrak{L}_0(\mathbb{R})$ , we need to show that if  $\Pi_2 u = 0$  and if  $\Pi_2 J_1 u = 0$ , then  $u = 0$  is the only possibility. We split  $\Pi_2 = \mathbf{I} + \Sigma_2$ , so that

$$\Sigma_2 u(t) = \sum_{j \in \mathbb{Z}^*} u(t + 2j),$$

where the two sides are to be understood liberally (compare with (10.2.3)). Then  $\Pi_2 u = 0$  is the same as  $u = -\Sigma_2 u$ , while  $\Pi_2 J_1 u = 0$  means that  $J_1 u = -\Sigma_2 J_1 u$ . Since  $J_1$  is an involution, we could write the latter as  $u = -J_1 \Sigma_2 J_1 u$ . We may need to be careful with the interpretation of the right-hand side, but let us not worry about that now. So, the two pieces of information we have about  $u \in \mathfrak{L}_0(\mathbb{R})$  is that  $u = -\Sigma_2 u$  and  $u = -J_1 \Sigma_2 J_1 u$ . We are free to combine them:

$$(11.3.1) \quad u = \Sigma_2 J_1 \Sigma_2 J_1 u \quad \text{and} \quad u = J_1 \Sigma_2 J_1 \Sigma_2 u.$$

If we write  $\mathbf{T}_1 := \Sigma_2 J_1$  and  $\mathbf{V}_1 := J_1 \Sigma_2$ , (11.3.1) maintains that  $u = \mathbf{T}_1^2 u$  and  $u = \mathbf{V}_1^2 u$ . The operator  $\mathbf{T}_1$  behaves like the transfer operator associated with the Gauss-type transformation  $\tau_1(x) = \{-1/x\}_2$  (see, e.g. (3.4.2)), but to get a precise fit we need to restrict our space of distributions to the symmetric standard interval  $I_1$ , and consider  $\mathfrak{L}(I_1)$ . Of course  $\mathbf{T}_1$  acts contractively on the space  $L^1(I_1)$  (see Proposition 3.4.1), but on the larger space  $\mathfrak{L}(I_1)$  it is no longer a norm contraction on the space (but it does define a bounded operator), see [17]. This is a serious complication, which is overcome only by a careful analysis of the action of the iterates of the transfer operator on the Hilbert kernel. We remark that on the interval  $I_1$ , the equality  $u = \mathbf{T}_1^2 u$  asks for  $u$  to be an “invariant configuration” in the state space  $\mathfrak{L}(I_1)$  for the composition square of the Gauss-type transformation. In the considerably simpler  $L^1(I_1)$  setting, this is the same as being a scalar multiple of the invariant measure (this observation uses ergodicity). From a functional analysis perspective, in the case of a finite mass invariant measure, ergodicity can be understood as the property that the given invariant measure is an extreme point in the convex body of all the invariant probability measures. In the case at hand, the absolutely continuous invariant measure is  $(1 - t^2)^{-1} dt$ , which is ergodic but has infinite mass, so it does not fit in the standard functional analysis interpretation. Then we still would know from ergodicity that the only possible solution to  $u = \mathbf{T}_1^2 u$  with  $u \in L^1(I_1)$  is the function  $u = 0$  (see, e.g. [16]). In this sense, the assertion that  $u = 0$  is the only possibility in the larger state space  $\mathfrak{L}(I_1)$  extends the standard ergodic theory. The analogue for a transformation without an indifferent fixed point would be the statement that the given invariant configuration is unique up to scalar multiples within the state space  $\mathfrak{L}(I_1)$ . As for the state space  $\mathfrak{L}(I_1)$ , we can think of this as arising from a mix of absolutely continuous signed densities of two types of particles, (i) point particles (represented by  $\delta_\xi$ ), and (ii) defocused particles (represented by  $\mathbf{H}\delta_\xi$ ). In the defocused case, we need to include source points  $\xi$  located outside the basic interval  $I_1$ ; if we would prefer to consider only  $\xi \in I_1$ , the Hilbert transform needs some slight modification to give the whole space  $\mathfrak{L}(I_1)$  in this manner.

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